1. Nikhil computes the sum of the first 10 positive integers, starting from 1. He then divides that sum by 5. What remainder does he get?

   Answer: 0

   Solution: We are computing the sum $1+2+\cdots+10$, which is equal to $\frac{10\cdot 11}{2} = 55$, and therefore the remainder upon division by 5 is $\boxed{0}$.

2. In class, starting at 8:00, Ava claps her hands once every 4 minutes, while Ella claps her hands once every 6 minutes. What is the smallest number of minutes after 8:00 such that both Ava and Ella clap their hands at the same time?

   Answer: 12

   Solution: We are trying to find the smallest number of minutes after which they can both meet up, which means we are trying to find the least positive integer that is divisible by both 4 and 6. In other words, we are looking for the least common multiple of 4 and 6. Since $4 = 2\cdot 2$ and $6 = 3\cdot 2$, we have that the lcm of 4 and 6 is $6\cdot 2 = \boxed{12}$.

3. A triangle has side lengths 3, 4, and 5. If all of the side lengths of the triangle are doubled, how many times larger is the area?

   Answer: 4

   Solution: Since $3^2+4^2 = 5^2$ (the side lengths satisfy the Pythagorean theorem), this is a right triangle with base 3, height 4, and hypotenuse 5. The area of this triangle is $\frac{1}{2}\cdot 3\cdot 4 = 6$. If the base and height are both doubled, the area becomes $\frac{1}{2}\cdot 6\cdot 8 = 24$. Thus, the area would be $\boxed{4}$ times larger.

4. There are 50 students in a room. Every student is wearing either 0, 1, or 2 shoes. An even number of the students are wearing exactly 1 shoe. Of the remaining students, exactly half of them have 2 shoes and half of them have 0 shoes. How many shoes are worn in total by the 50 students?

   Answer: 50

   Solution: Suppose there are x students wearing 1 shoe. Then $\frac{50-x}{2}$ students are wearing 2 shoes and $\frac{50-x}{2}$ students are wearing 0 shoes. Hence, the students are wearing a total of

$$\begin{aligned} 1\cdot x + 2\cdot\frac{50-x}{2} + 0\cdot\frac{50-x}{2} &= x + (50-x) \\ &= \boxed{50} \end{aligned}$$

   shoes.

5. What is the value of $-2+4-6+8-\cdots+8088$?

   Answer: 4044

   Solution:

   We pair consecutive terms in the sum and notice that $-2+4 = -4+6 = \cdots = -8086+8088 = 2$. There are $\frac{8088}{2} = 4044$ terms in the sum, so we have $\frac{4044}{2} = 2022$ pairs that each sum to 2, giving us a total of $\boxed{4044}$.

6. Suppose Lauren has 2 cats and 2 dogs. If she chooses 2 of the 4 pets uniformly at random, what is the probability that the 2 chosen pets are either both cats or both dogs?

Answer: $\mathbf{\frac{1}{3}}$

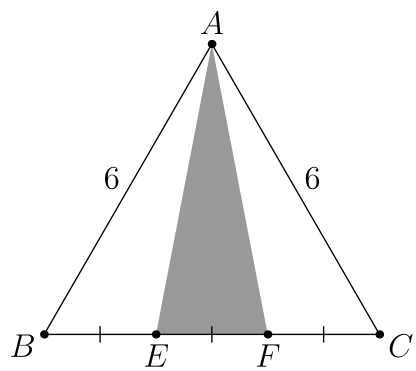
Solution:

There is 1 way for both cats to be chosen and 1 way to for both dogs to be chosen. Since there are 4 pets, there are $\binom{4}{2} = 6$ total ways to choose 2 pets. Thus, the answer is $\frac{2}{6} = \boxed{\frac{1}{3}}$.

7. Let triangle $\triangle ABC$ be equilateral with side length 6. Points E and F lie on $\overline{BC}$ such that E is closer to B than it is to C and F is closer to C than it is to B. If $BE = EF = FC$, what is the area of triangle $\triangle AFE$?

   Answer: $\mathbf{3\sqrt{3}}$

   Solution: We build the following diagram, where we want to compute the area of the shaded region.



   Notice $BC = BE + EF + FC = 3EF$, so $FE = \frac{BC}{3} = \frac{6}{3} = 2$. Thus, the area of triangle $\triangle AFE$ is $\frac{1}{3}$ the area of triangle $\triangle ABC$. Using the formula for the area of an equilateral triangle, which is $s^2 \cdot \frac{\sqrt{3}}{4}$ for side length s, we compute that the area of triangle $\triangle AFE = \boxed{3\sqrt{3}}$.

8. The two equations $x^2 + ax - 4 = 0$ and $x^2 - 4x + a = 0$ share exactly one common solution for x. Compute the value of a.

   Answer: $\mathbf{3}$

   Solution: Let r be the common solution. Then we must have that $r^2 + ar - 4 = 0 = r^2 - 4r + a$, so moving all the r-terms to one side gives that $(a + 4)r = a + 4$. Thus, either $a + 4 = 0$, or $r = 1$. If $a + 4 = 0$, then $a = -4$, but then in this case the two quadratic equations will be the same, and we can show there will be a common solution distinct from r. Thus, we must have that $r = 1$, so $1 + a - 4 = 0$, giving that $a = \boxed{3}$.

9. At Shreymart, Shreyas sells apples at a price c. A customer who buys n apples pays nc dollars, rounded to the nearest integer, where we always round up if the cost ends in .5. For example, if the cost of the apples is 4.2 dollars, a customer pays 4 dollars. Similarly, if the cost of the apples is 4.5 dollars, a customer pays 5 dollars. If Justin buys 7 apples for 3 dollars and 4 apples for 1 dollar, how many dollars should he pay for 20 apples?

   Answer: $\mathbf{7}$

   Solution: The problem conditions give that

$$\frac{2.5}{7} \leq c < \frac{3.5}{7},$$

and

$$\frac{0.5}{4} \leq c < \frac{1.5}{4}.$$

Combining these conditions together gives

$$0.357 < c < 0.375.$$

Thus,

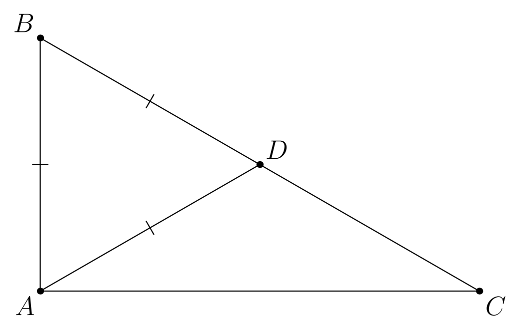
$$7.14 < 20c < 7.5,$$

so $20c$ rounds to $\boxed{7}$.

10. In triangle $\triangle ABC$, the angle trisector of $\angle BAC$ closer to $\overline{AC}$ than $\overline{AB}$ intersects $\overline{BC}$ at D. Given that triangle $\triangle ABD$ is equilateral with area 1, compute the area of triangle $\triangle ABC$.

    Answer: 2

    Solution: We have the following diagram.



Let $AB = BD = DA = x$. Then note that $\angle DAC = 30°$ and $\angle ADC = 120°$, so $\angle ACD = 30°$. As $DA = x$, we have $DC = x$. Since triangle $\triangle ABC$ is right, with $2x$ as the hypotenuse length, its other leg has length $x\sqrt{3}$, so it has area $\frac{x^2\sqrt{3}}{2}$. The area of an equilateral triangle with side length x is $\frac{x^2\sqrt{3}}{4}$. We are given that $\frac{x^2\sqrt{3}}{4} = 1$, so $\frac{x^2\sqrt{3}}{2} = \boxed{2}$.

11. Wanda lists out all the primes less than 100 for which the last digit of that prime equals the last digit of that prime's square. For instance, 71 is in Wanda's list because its square, 5041, also has 1 as its last digit. What is the product of the last digits of all the primes in Wanda's list?

    Answer: 5

    Solution: The main idea is that the only primes whose last digit equals the last digit of their square are primes that end in either 5 or 1. Therefore, we don't actually have to find these primes, because when we multiply all the last digits of these primes together, we will get $5 \times 1 \times 1 \times \cdots \times 1 = \boxed{5}$.

12. How many ways are there to arrange the letters of `SUSBUS` such that `SUS` appears as a contiguous substring? For example, `SUSBUS` and `USSUSB` are both valid arrangements, but `SUBSSU` is not.

    Answer: 22

    Solution: We can do casework on where the `SUS` appears.

    - `SUS___`
      There are $3! = 6$ cases here since we can arrange the `B`, `U`, `S` in any order.

- `_SUS__`
  There are $3! = 6$ cases here since we can arrange the `B`, `U`, `S` in any order.
- `__SUS_`
  There are $3! = 6$ cases here since we can arrange the `B`, `U`, `S` in any order.
- `___SUS`
  There are $3! = 6$ cases here since we can arrange the `B`, `U`, `S` in any order.

However, we have overcounted, since we have counted `SUSUSB` and `BSUSUS` twice. Thus, if we subtract 2 from the answer above, we will get the correct answer. Therefore the answer is

$$24 - 2 = \boxed{22}.$$

13. Suppose that x and y are integers such that $x \geq 5$, $y \geq 3$, and $\sqrt{x-5} + \sqrt{y-3} = \sqrt{x+y}$. Compute the maximum possible value of xy.

Answer: 114

Solution: Squaring both sides gives

$$(x-5) + (y-3) + 2\sqrt{(x-5)(y-3)} = x + y,$$

so

$$\sqrt{(x-5)(y-3)} = 4,$$

and hence

$$(x-5)(y-3) = 16.$$

Now, 16 can only be factored in 5 ways:

$$(1, 16),\ (2, 8),\ (4, 4),\ (8, 2), \text{ and } (16, 1).$$

We can test all of them to see that $x - 5 = 1$ and $y - 3 = 16$ gives the maximum possible value of $\boxed{114}$ for $x = 6$ and $y = 19$.

14. What is the largest integer k divisible by 14 such that $x^2 - 100x + k = 0$ has two distinct integer roots?

Answer: 2464

Solution: The key here is to recognize that the sum of the solutions to this quadratic is 100. Thus, we must find integers a and b such that $a + b = 100$ and $a \cdot b = k$, for some k divisible by 14. To list out all the possibilities, we can start with a as 14 and b as 86, and then sequentially add 14 to a and subtract 14 from b. This way, $a + b$ will always equal 100, and there will always be a factor of 14 in $a \cdot b$. Note that we disregard negative values for a and b, because if both values were negative, then the sum of the two would not be 100, and if one of the values were negative, then k would not be positive, so it wouldn't be the largest possible integer. Hence, the possibilities are

$$(a, b) \in \{(14, 86), (28, 72), (42, 58), (56, 44), (70, 30), (84, 16), (98, 2)\}.$$

We know that the product will be maximized when the integers are as close to 50 as possible, so our answer is $56 \cdot 44 = \boxed{2464}$.

15. What is the sum of the first 16 positive integers whose digits consist of only 0s and 1s?

Answer: 18888

Solution: We write out the first 16 binary numbers, which are $\overline{ABCD}_2$ and 10000 for choices of binary digits A, B, C, D, keeping in mind to exclude the case where all of them are 0s. The sum of the $\overline{ABCD}_2$s when read as base-ten numbers can be computed by looking at each digit individually: it is a 1 for $2^3 = 8$ of the $\overline{ABCD}_2$s, thus it contributes 8 in that place. Adding in 10000 gives a total of $\boxed{18888}$.

16. Jonathan and Ajit are flipping two unfair coins. Jonathan's coin lands on heads with probability $\frac{1}{20}$ while Ajit's coin lands on heads with probability $\frac{1}{22}$. Each year, they flip their coins at the same time, independently of their previous flips. Compute the probability that Jonathan's coin lands on heads strictly before Ajit's coin does.

Answer: $\mathbf{\frac{21}{41}}$

Solution: Let W be the event that Jonathan's coin lands on heads strictly before Ajit's coin does. Consider the first year. If Ajit's coin lands on heads, then we know that W cannot occur. Otherwise, if Jonathan's coin lands on heads, then we know that W does occur. Finally, if neither coin lands on heads, then W occurs with probability $P(W)$ since we are back where we started. Thus, we have the equation

$$P(W) = \frac{1}{20} \cdot \frac{21}{22} + \frac{19}{20} \cdot \frac{21}{22} P(W).$$

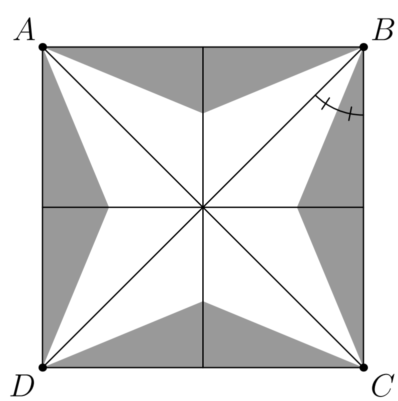
Solving this equation, we find that

$$P(W) = \boxed{\frac{21}{41}}.$$

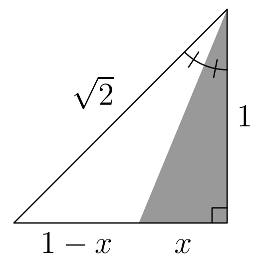
17. A point is chosen uniformly at random in square $ABCD$. What is the probability that it is closer to one of the 4 sides than to one of the 2 diagonals?

Answer: $\mathbf{\sqrt{2} - 1}$

Solution: The key to the solution is to divide square $ABCD$ into eight congruent triangles, as follows.



We would like to compute the ratio of the shaded area to the total area, so by symmetry, it suffices to compute this ratio within one of the eight shaded triangles. As such, we take $AB = 2$ without loss of generality and write out one of the triangles as follows.



Now, from the Angle bisector theorem, we write

$$\frac{x}{1} = \frac{1-x}{\sqrt{2}},$$

which gives $1 - x = x\sqrt{2}$ and hence

$$x = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1.$$

Thus, the ratio between the desired areas is

$$\frac{\frac{1}{2}\cdot 1 \cdot x}{\frac{1}{2}\cdot 1\cdot 1} = x = \boxed{\sqrt{2}-1},$$

which is what we wanted.

18. Two integers are coprime if they share no common positive factors other than 1. For example, 3 and 5 are coprime because their only common factor is 1. Compute the sum of all positive integers that are coprime to 198 and less than 198.

Answer: 5940

Solution: Recall that, for all positive integers n, $\gcd(n, 198) = \gcd(198 - n, 198)$. We can use this idea to write the sum S forwards and backwards:

$$S = 1 + 5 + 7 + \cdots + 197$$

$$S = 197 + 193 + 191 + \cdots + 1.$$

Adding the sums, we get

$$2S = 198 + 198 + 198 + \cdots + 198.$$

To find the number of 198s on the right hand side, we need to find the number of positive integers which are less than 198 and coprime to 198. This is given by Euler's totient formula: as $198 = 2\cdot 3^2 \cdot 11$, we have $\phi(198) = 198 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{10}{11} = 60$. Thus, $2S = 198 \cdot 60$, so $S = 198 \cdot 30 = \boxed{5940}$.

19. Sumith lists out the positive integer factors of 12 in a line, writing them out in increasing order as $1, 2, 3, 4, 6, 12$. Luke, being the mischievious person he is, writes down a permutation of those factors and lists it right under Sumith's as $a_1, a_2, a_3, a_4, a_5, a_6$. Luke then calculates

$$\gcd(a_1, 2a_2, 3a_3, 4a_4, 6a_5, 12a_6).$$

Given that Luke's result is greater than 1, how many possible permutations could he have written?

Answer: 308

Solution: Note that the gcd must be divisible by 2 or 3. We proceed with the Principle of Inclusion-Exclusion.

For the gcd to be divisible by 2, the 1 and 3 in Sumith's list must both be matched with Luke's even numbers. Since there are 4 total even factors of 12, there are $4 \cdot 3 = 12$ ways of doing so. There's no restriction on the other four of Luke and Sumith's numbers, so the total number of ways is $12 \cdot 4! = 288$.

For the gcd to be divisible by 3, note that there are 3 factors divisible by 3. Hence, each of Sumith's numbers not divisible by 3 must be matched with 1 of Luke's numbers divisible by 3. There are $(3!)^2 = 36$ ways for this.

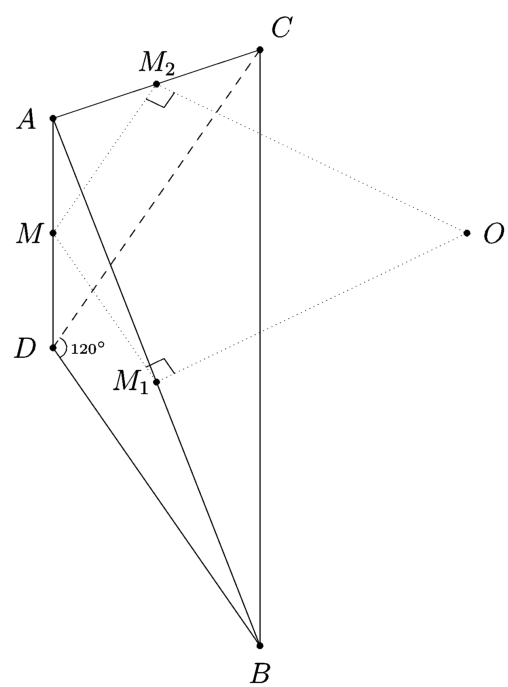
Finally, we subtract the number of ways the gcd is divisible by 6. There are 2 ways of picking what Sumith's 1 matches with: the 6 or 12. Next, Sumith's 2 and 4 must be paired with Luke's 2 other factors that are divisible by 3, and there are 2 ways of ordering those. There are then 2 ways of picking what Sumith's 3 matches with: the 2 or the 4. Finally, Sumith's 6 and 12 must be paired with Luke's remaining 2 numbers, with a total of 2 possibilities. Thus, the number of ways is $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

Hence, $288 + 36 - 16 = \boxed{308}$.

20. Tetrahedron $ABCD$ is drawn such that $DA = DB = DC = 2$, $\angle ADB = \angle ADC = 90^\circ$, and $\angle BDC = 120^\circ$. Compute the radius of the sphere that passes through A, B, C, and D.

Answer: $\sqrt{5}$

Solution: We have the following diagram:



Let M_1 be the midpoint of $\overline{AB}$ and M_2 be the midpoint of $\overline{AC}$. Let l_1 be the line through M_1 perpendicular to plane ABD. Let l_2 be the line through M_2 perpendicular to plane ACD. The center of the sphere O is equidistant from A, B, C, and D. So, O must lie on lines l_1 and l_2, and therefore is the intersection of those two lines. Let M be the midpoint of $\overline{AD}$. We know $\angle M_1MM_2 = 120^\circ$ and $\angle OM_1M = 90^\circ = \angle OM_2M$. These facts, along with the fact that the plane M_1MM_2 is parallel to the plane BDC, show that triangle $\triangle MM_1O$ and triangle $\triangle MM_2O$ are $30^\circ - 60^\circ - 90^\circ$ triangles. So, since $MM_1 = MM_2 = 1$, $OM_1 = OM_2 = \sqrt{3}$. We also can calculate $DM_1 = DM_2 = \sqrt{2}$. Since $\overline{DM_1}$ is perpendicular to $\overline{M_1O}$ and $\overline{DM_2}$ is perpendicular to $\overline{M_2O}$, we can use the Pythagorean Theorem to compute our answer: $\sqrt{\sqrt{2}^2 + \sqrt{3}^2} = \boxed{\sqrt{5}}$.