1. If x is 20% of 23 and y is 23% of 20, compute $\frac{x}{y}$.

   Answer: 1

   Solution: We see mathematically that $0.20 \cdot 23 = 0.23 \cdot 20$. The first expression is x and the second expression is y so this means $x = y$. This means $\frac{x}{y} = \boxed{1}$.

2. Pablo wants to eat a banana, a mango, and a tangerine, one at a time. How many ways can he choose the order to eat the three fruits?

   Answer: 6

   Solution: There are 3 ways to eat the first fruit, 2 for the second, and 1 for the last, with a total of $3 \cdot 2 \cdot 1 = \boxed{6}$ ways. One can also alternatively list out all of the arrangements.

3. Let a, b, and c be 3 positive integers. If $a + \frac{b}{c} = \frac{11}{6}$, what is the minimum value of $a + b + c$?

   Answer: 12

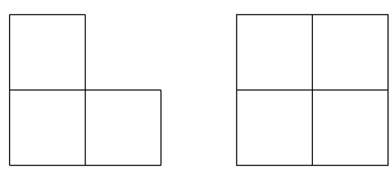
   Solution: First of all, a must be 1 since otherwise the sum is over 2 which is greater than $\frac{11}{6}$. Then we know that $\frac{b}{c} = \frac{5}{6}$. To minimize the value of $a + b + c$, we write $\frac{b}{c}$ in the simplest form, which is $\frac{5}{6}$. Thus, $\frac{11}{6} = 1 + \frac{5}{6}$ is the one with the lowest sum of a, b, and c. The sum of $a + b + c$ is $1 + 5 + 6 = \boxed{12}$.

4. A rectangle has an area of 12. If all of its sidelengths are increased by 2, its area becomes 32. What is the perimeter of the original rectangle?

   Answer: 16

   Solution: Let a and b be the sidelengths of the rectangle. We have that $ab = 12$ and $(a+2)(b+2) = ab + 2(a + b) + 4 = 32$. Substituting, we get that the perimeter is $2(a + b) = 32 - 4 - ab = 32 - 4 - 12 = \boxed{16}$. We can verify this construction by noticing that the original rectangle has length 6 and width 2.

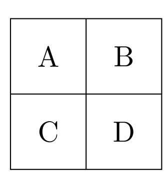
5. Rohit is trying to build a 3-dimensional model by using several cubes of the same size. The model's front view and top view are shown below. Suppose that every cube on the upper layer is directly above a cube on the lower layer and the rotations are considered distinct. Compute the total number of different ways to form this model.



Front View Top View

   Answer: 3

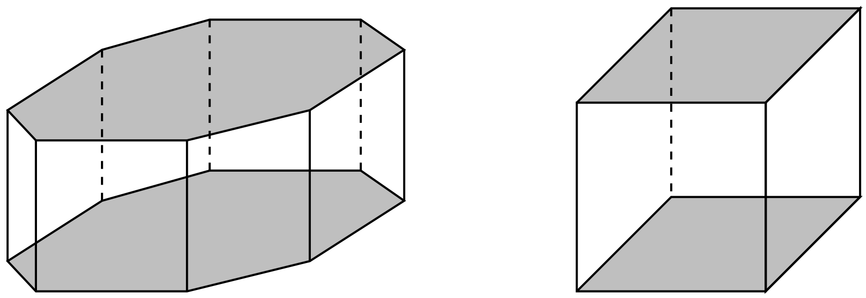
   Solution:



Top Layer From the Top View

Based on the front view and the top view, we know that this model has two layers and the bottom layer must have 4 cubes. For the top layer, if we see from the top, there are 3 different possibilities: only A, only C, or both A and C. So in total, there are $\boxed{3}$ different ways.

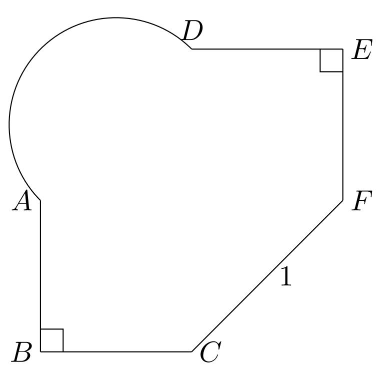
6. Priscilla has three octagonal prisms and two cubes, none of which are touching each other. If she chooses a face from these five objects in an independent and uniformly random manner, what is the probability the chosen face belongs to a cube? (One octagonal prism and cube are shown below.)



Answer: $\mathbf{\frac{2}{7}}$

Solution: There are six faces on each cube and then eight side faces and two bases on an octagonal prism. Therefore, there are $2 \cdot 6 = 12$ faces on both of the cubes and $2 \cdot 6 + 3 \cdot (8 + 2) = 12 + 30 = 42$ total faces on all five objects. Thus, the probability that a randomly chosen face is on a cube is $\frac{12}{42} = \boxed{\frac{2}{7}}$.

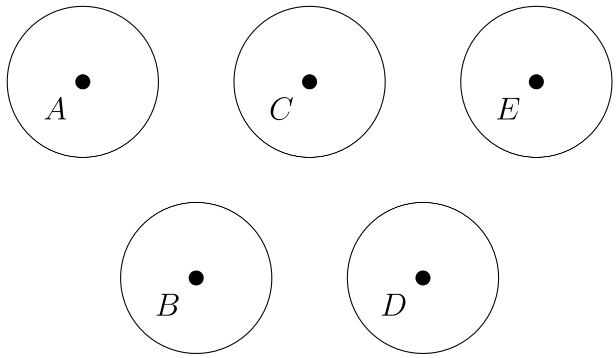
7. Let triangle $\triangle ABC$ and triangle $\triangle DEF$ be two congruent isosceles right triangles where line segments $\overline{AC}$ and $\overline{DF}$ are their respective hypotenuses. Connecting a line segment $\overline{CF}$ gives us a square $ACFD$ but with missing line segments $\overline{AC}$, $\overline{AD}$, and $\overline{DF}$. Instead, A and D are connected by an arc defined by the semicircle with endpoints A and D. If $CF = 1$, what is the perimeter of the whole shape $ABCFED$ (shown on the next page)?



Answer: $\mathbf{2\sqrt{2} + 1 + \frac{\pi}{2}}$

Solution: Since $CF = 1$ then $AC = DF = 1$ and the semicircle arc $\overset{\frown}{AD}$ has a diameter of 1. So the arc $\overset{\frown}{AD} = \frac{\pi}{2}$. Since triangle $\triangle ABC$ and triangle $\triangle DEF$ are 45-45-90 right triangles, $AB = BC = DE = EF = \frac{1}{\sqrt{2}}$. So the perimeter is: $\frac{\pi}{2} + 1 + \frac{4}{\sqrt{2}} = \boxed{\frac{\pi}{2} + 1 + 2\sqrt{2}}$.

8. There are two moles that live underground, and there are five circular holes that the moles can hop out of. The five holes are positioned as shown in the diagram below, where A, B, C, D, and E are the centers of the circles, $AE = 30$ cm, and congruent triangles $\triangle ABC$, $\triangle CBD$, and $\triangle CDE$ are equilateral. The two moles randomly choose exactly two of the five holes, hop out of the two chosen holes, and hop back in. What is the probability that the holes that the two moles hop out of have centers that are exactly 15 cm apart?



Answer: $\mathbf{\frac{7}{10}}$

Solution: Note that $AB = AC = BC = BD = CD = CE = DE$. Because $AC + CE = AE = 30$ cm, we must have $AC = 15$ cm.

The total number of possibilities for the two holes to be chosen is $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$. By noting that the side lengths of the three equilateral triangles are 15 cm, we find that there are 7 possibilities where the two holes have centers that are exactly 15 cm, which are the endpoints of the edges $AB, AC, BC, BD, CD, CE, DE$. Thus, the probability that the moles hop in and out of holes where the centers are 15 cm apart is $\boxed{\frac{7}{10}}$.

9. Carson is planning a trip for n people. Let x be the number of cars that will be used and y be the number of people per car. What is the smallest value of n such that there are exactly 3 possibilities for x and y so that y is an integer, $x < y$, and exactly one person is left without a car?

Answer: 13

Solution:
Solution 1: We have $n - 1 = xy$, so we require $n - 1$ to be the smallest integer with 6 factors. If $n - 1$ has 6 factors, then either $n - 1 = p^5$ for some prime p or $n - 1 = p^2 q$ for primes p and q. We find the minimum value in the second case with $p = 2$ and $q = 3$, yielding $n - 1 = 12$ and $n = \boxed{13}$.

Solution 2: We have $xy = n - 1$ (the minus one is from the fact that there's exactly one person without a car). Since there are 3 combinations of x and y, this results in $n - 1$ having 6 factors, so that x can be any of the 3 smallest factors, and y being the corresponding factor to x. Since we want the smallest value of n (and therefore $n - 1$), we can see that 12 is the smallest number with exactly 6 factors, $\{1, 2, 3, 4, 6, 12\}$. Therefore, $xy = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 12$, so $n - 1 = 12$, meaning $n = \boxed{13}$.

10. Iris is eating an ice cream cone, which consists of a hemisphere of ice cream with radius $r > 0$ on top of a cone with height 12 and also radius r. Iris is a slow eater, so after eating one-third

of the ice cream, she notices that the rest of the ice cream has melted and completely filled the cone. Assuming the ice cream did not change volume after it melted, what is the value of r?

Answer: 9

Solution: The volume of the ice cream (the hemisphere) is $\frac{2}{3}\pi r^3$ and the volume of the cone is $\frac{1}{3}\pi r^2(12)$. Since one-third of the ice cream was eaten, two-thirds of the ice cream's volume is equal to the cone's volume, so we have $\frac{2}{3} \cdot \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^2(12)$. After solving this equation, we get $r = 0$ or 9. Neglecting the non-positive root, the radius of the cone is $\boxed{9}$.

11. As Natasha begins eating brunch between 11:30 AM and 12 PM, she notes that the smaller angle between the minute and hour hand of the clock is 27 degrees. What is the number of degrees in the smaller angle between the minute and hour hand when Natasha finishes eating brunch 20 minutes later?

Answer: 83

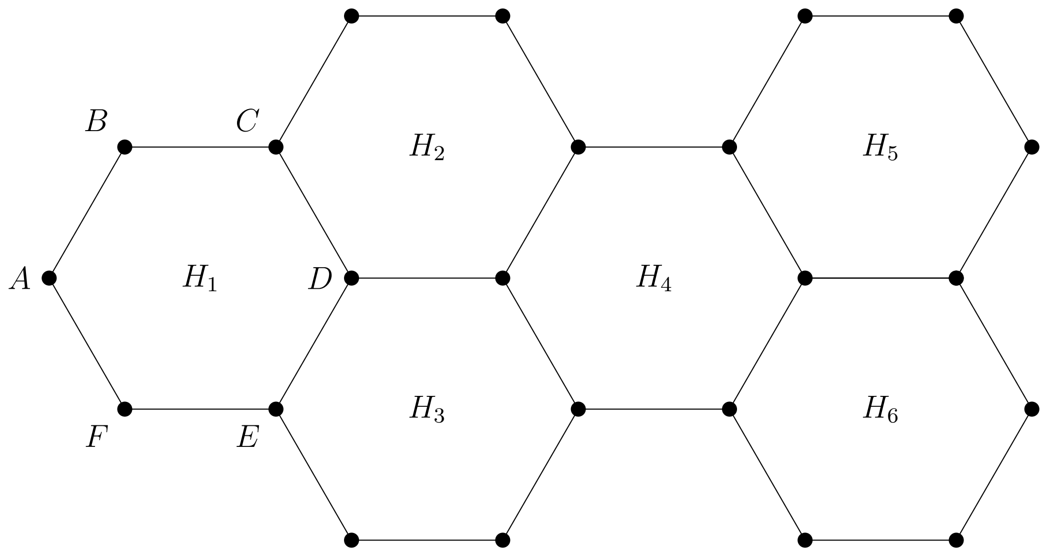
Solution: The minute hand will move 120 degrees in 20 minutes, and the hour hand will move 10 degrees. So the minute hand will move 110 more degrees than the hour hand, and will move past it, and so the angle after 20 minutes is $110 - 27 = \boxed{83}$ degrees.

12. On a regular hexagon $ABCDEF$, Luke the frog starts at point A; there is food on points C and E and there are crocodiles on points B and D. When Luke is on a point, he hops to any of the five other vertices with equal probability. What is the probability that Luke will visit both of the points with food before visiting any of the crocodiles?

Answer: $\frac{1}{6}$

Solution: A way of representing the situation is by considering infinite strings where each character is generated randomly from A, B, C, D, E, and F, but cannot be equal to the previous character. The frog is successful when both C and E are generated before both the B and the D, and the frog fails otherwise. By symmetry, we have to calculate the probability that a randomly selected permutation of $BCDE$ will have the C and the E each before the B and D. As C and E occupy the first two characters while B and D occupy the last two characters, there are 4 desired permutations out of the 24 total, yielding a probability of $\frac{4}{24} = \boxed{\frac{1}{6}}$.

13. 2023 regular unit hexagons are arranged in a tessellating lattice, as follows. The first hexagon $ABCDEF$ (with vertices in clockwise order) has leftmost vertex A at the origin, and hexagons H_2 and H_3 share edges $\overline{CD}$ and $\overline{DE}$ with hexagon H_1, respectively. Hexagon H_4 shares edges with both hexagons H_2 and H_3, and hexagons H_5 and H_6 are constructed similarly to hexagons H_2 and H_3. Hexagons H_7 to H_{2022} are constructed following the pattern of hexagons H_4, H_5, H_6. Finally, hexagon H_{2023} is constructed, sharing an edge with both hexagons H_{2021} and H_{2022}. Compute the perimeter of the resulting figure.



Answer: 5398

Solution: Note that $2023 = 674 \cdot 3 + 1$. So there are 674 copies of the first three hexagons with one single hexagon H_{2023} left. Each copy except the first three contributes 8 to the perimeter. The first three hexagons has 2 extra unit lengths contributed by $\overline{AB}$ and $\overline{AF}$. The hexagon H_{2023} contributes 4 extra unit lengths. So the entire perimeter is $8 \cdot 674 + 2 + 4 = \boxed{5398}$.

14. Aditya's favorite number is a positive two-digit integer. Aditya sums the integers from 5 to his favorite number, inclusive. Then, he sums the next 12 consecutive integers starting after his favorite number. If the two sums are consecutive integers and the second sum is greater than the first sum, what is Aditya's favorite number?

Answer: 29

Solution: Let n be Aditya's favorite number. The first sum is equal to $\frac{n(n+1)}{2} - \frac{4 \cdot 5}{2} = \frac{1}{2}n^2 + \frac{1}{2}n - 10$, and the second sum is equal to $12n + \frac{12 \cdot 13}{2} = 12n + 78$. The first sum is exactly one less than the second sum, so $\frac{1}{2}n^2 + \frac{1}{2}n - 10 = (12n + 78) - 1$. Rearranging the terms to form a quadratic gives $\frac{1}{2}n^2 - \frac{23}{2}n - 87 = 0$.

Solving this equation via the quadratic formula, we get that n is either 29 or -6. Discarding the negative root, we conclude that Aditya's favorite number is $\boxed{29}$.

15. The 100^{th} anniversary of BMT will fall in the year 2112, which is a palindromic year. Compute the sum of all years from 0000 to 9999, inclusive, that are palindromic when written out as four-digit numbers (including leading zeros). Examples include 2002, 1991, and 0110.

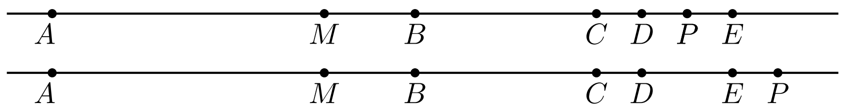
Answer: 499950

Solution: All palindromes with four digits are of the form $abba$, where $0 \leq a, b \leq 9$. The contributions for each digit in each place need to be counted 10 times, as that is the number of palindromes that digit appears in that place. Hence, the sum is $10 \cdot 1001(0 + 1 + 2 + \cdots + 9) + 10 \cdot 110(0 + 1 + 2 + \cdots + 9) = 11110 \cdot 45 = \boxed{499950}$.

16. Points A, B, C, D, and E lie on line r, in that order, such that $DE = 2DC$ and $AB = 2BC$. Let M be the midpoint of segment $\overline{AC}$. Finally, let point P lie on r such that $PE = x$. If $AB = 8x$, $ME = 9x$, and $AP = 112$, compute the sum of the two possible values of CD.

Answer: 15

Solution: It is a good idea to draw a picture. But first, we must notice that there are two possibilities for the location of point P: it can either lie after E, or between D and E.



Analyzing the given information,

$$\begin{aligned} AB &= \frac{2}{3}AC \\ 8x &= \frac{2}{3}AC \\ AC &= 12x, \end{aligned}$$

so

$$\begin{aligned} AM = MC &= \frac{1}{2}AC \\ AM = MC &= \frac{12x}{2} \\ AM = MC &= 6x \end{aligned}$$

and

$$\begin{aligned} ME &= MD + DE \\ 9x &= MD + 2x \\ MD &= 7x. \end{aligned}$$

We conclude that

$$\begin{aligned} MD &= MC + CD \\ 7x &= 6x + CD \\ CD &= x. \end{aligned}$$

Now we must find the value of x by using the information that $AP = 112$. But since there are two possible locations for point P, we will find two values for x:

For the case represented by the first picture,

$$\begin{aligned} AP &= AC + CD + DP \\ AP &= AC + CD + (DE - PE) \\ 112 &= 12x + x + (2x - x) \\ 14x &= 112 \\ x &= 8. \end{aligned}$$

For the case represented by the second picture,

$$\begin{aligned} AP &= AC + CD + DE + EP \\ 112 &= 12x + x + 2x + x \\ 16x &= 112 \\ x &= 7. \end{aligned}$$

Therefore, the sum of the two possible values of CD is given by $8 + 7 = \boxed{15}$.

17. A parabola $y = x^2$ in the xy-plane is rotated 180° about a point (a, b). The resulting parabola has roots at $x = 40$ and $x = 48$. Compute $a + b$.

Answer: 30

Solution: When we rotate a parabola by 180° about a point, we will get a parabola of the form $-x^2 + cx + d$. Since we know the roots of the parabola, we get that the parabola is $-(x-40)(x-48)$, which has its vertex at $(44, 16)$. The point of rotation would be the midpoint of the segment between the vertices of the two parabolas, which is $(22, 8)$, and so our answer is $22 + 8 = \boxed{30}$.

18. Susan has a standard die with values 1 to 6. She plays a game where every time she rolls the die, she permanently increases the value on the top face by 1. What is the probability that, after she rolls her die 3 times, there is a face on it with a value of at least 7?

Answer: $\frac{35}{72}$

Solution: Let us roll the die 3 times and change the value of the die at the end of the process. There are 3 disjoint cases as Susan either rolls three 4s, two 5s and another number which is neither 5 nor 6, three 5s, or at least one 6.

The probability Susan rolls three 4s or three 5s is each $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$. The probability she rolls two 5s and a number at most 4 is $\frac{(4\cdot 3)}{6^3} = \frac{12}{216}$ since there are 4 ways to choose the other number and 3 ways to choose its position. Finally, the probability that Susan rolls at least one 6 is 1 minus the probability that Susan rolls no 6s which occurs with probability $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$. Thus, the last probability is $1 - \frac{125}{216} = \frac{91}{216}$, so the entire probability is $2 \cdot \frac{1}{216} + \frac{12}{216} + \frac{91}{216} = \frac{105}{216} = \boxed{\frac{35}{72}}$.

19. Let N be a 6-digit number satisfying the property that the average value of the digits of N^4 is 5. Compute the sum of the digits of N^4.

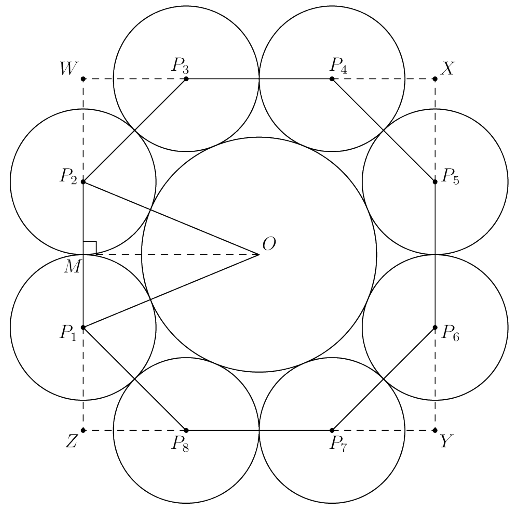
Answer: 115

Solution: Note that N^4 either has 21, 22, 23, or 24 digits, which would give digit sums of 105, 110, 115, or 120, respectively. If N^4 had a sum of digits of 105 or 120, that would imply that N^4 is divisible by 3 but not 9, which is not possible since N^4 is a perfect square. If N^4 had a sum of digits of 110, that would imply that N^4 would leave a remainder of 2 when divided by 3, which is not possible since N^4 is a perfect square. Therefore, N^4 must have a digit sum of $\boxed{115}$. (For correctness, one such value of N that satisfies the property is 510932.)

20. Let O_1, O_2,..., O_8 be circles of radius 1 such that O_1 is externally tangent to O_8 and O_2 but no other circles, O_2 is externally tangent to O_1 and O_3 but no other circles, and so on. Let C be a circle that is externally tangent to each of $O_1, O_2, O_3, \ldots, O_8$. Compute the radius of C.

Answer: $\boldsymbol{\sqrt{4+2\sqrt{2}}-1}$

Solution:



Let O be the center of C, and let P_i be the center of O_i for $1 \le i \le 8$. A regular octagon with sidelength of 2 is constructed by connecting line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, $\overline{P_4P_5}$, $\overline{P_5P_6}$, $\overline{P_6P_7}$, $\overline{P_7P_8}$, and $\overline{P_8P_1}$.

Let W, X, Y, and Z be the respective intersections of lines $\overline{P_1P_2}$ and $\overline{P_3P_4}$, $\overline{P_3P_4}$ and $\overline{P_5P_6}$, $\overline{P_5P_6}$ and $\overline{P_7P_8}$, $\overline{P_7P_8}$ and $\overline{P_1P_2}$. We can see that triangles $\triangle P_1ZP_8$, $\triangle P_2WP_3$, $\triangle P_4XP_5$, and $\triangle P_6YP_7$ are $45-45-90$ triangles, so it follows that $WXYZ$ is a square. Since triangle $\triangle P_1ZP_8$ is a 45-45-90 triangle with hypotenuse 2, its legs $ZP_1 = ZP_8 = \sqrt{2}$. Thus, the sidelength of square $WXYZ$ is $2\sqrt{2}+2$.

Let M be the midpoint of line segment $\overline{P_1P_2}$. We have $P_1M = 1$ and $OM = \frac{WX}{2} = 1+\sqrt{2}$. By the Pythagorean Theorem on triangle $\triangle P_1MO$, $P_1O = \sqrt{1^2+(1+\sqrt{2})^2} = \sqrt{4+2\sqrt{2}}$. Finally, to get the radius of C, we let K be the point of tangency between O_1 and C. We have the radius of C is $P_1O - P_1K = \boxed{\sqrt{4+2\sqrt{2}}-1}$.