1. Round 1

1. $$8051 = 8100 - 49 = (90 - 7)(90 + 7)$$

2. Recall that $1 = \log_x x$. Thus, we have
$$\begin{aligned}
[\log_{xyz}(x^z)][1 + \log_x y + \log_x z] &= [\log_{xyz}(x^z)][\log_x x + \log_x y + \log_x z] \\
&= [\log_{xyz}(x^z)][\log_x xyz] \\
&= [\log_x xyz][\log_{xyz}(x^z)] \\
&= \log_x x^z \\
&= z
\end{aligned}$$

3. The number of sequences with no A's is 3^6; The number of sequences with no U's is 3^6.The number of sequences with no A's or U's is 2^6. The total number of sequences is 4^6. Thus by inclusion-exclusion, we have that the number of sequences with no A's or U's is
$$2 * 3^6 - 2^6 = 1394.$$

4. We want to have $17^n + n$ divisible by 29, which means $17^n \equiv -n \bmod 29$. We have $17^1 \equiv -12$, $17^2 = 289 \equiv -1 \bmod 29$, $17^3 \equiv -17$, and $17^4 \equiv 1 = -28 \bmod 29$. Thus our possibilities for $n < 29$ having $17^n \equiv -n$ are 12, 1, 17 or 28. But $17^{12} \equiv 1$, $17^1 \equiv 17$, and $17^{17} \equiv 17$. However, $17^{28} \equiv 1 = -28$. Thus, our smallest positive solution is $n = 28$.

5. Label the angle between the side of length b and side of length c as θ, and label the width of rectangle R as y. We then have
$$\cos\theta = \frac{b}{c}$$
$$\cos\theta = \frac{y}{b}$$
Setting the two equal yields
$$\frac{b}{c} = \frac{y}{b} \quad \Rightarrow \quad yc = b^2$$
Therefore, the area of the smaller rectangle is $b^2 = 1024$.

6. In general, there are $\binom{n}{k}2^{n-k}$ k-cubes in an n-cube (where $k \le n$). The vertices that define an n-cube can be thought of as the set of strings $(x_1, ..., x_n)$ where each $x_i \in \{0, 1\}$. Then, a k-cube is a set of points fixing all but k of the x_is, and letting the others freely vary. There are $\binom{n}{k}$ ways to choose which points to fix, and 2^{n-k} ways to decide what to fix them at. Thus, we have $\binom{n}{k}2^{n-k}$ k-cubes in an n-cube. When $n = 5$ and $k = 3$, we have $\binom{5}{3}2^2 = 40$ 3-cubes in a 5-cube.

2. Round 2

1. Let's look at the probability of one bin being empty. The probability is simply $\left(1 - \frac{1}{n}\right)^k$, since all balls avoid this bin. Using linearity of expectation, our final answer is simply $n \cdot \left(\frac{n-1}{n}\right)^k = 625/216$.

2. Using $221 = 13 \cdot 17$, we can take advantage of the fractal nature of Pascal's triangle to calculate the binomial coefficient modulo a prime. We get that $\binom{150}{20} \equiv \binom{\lfloor \frac{150}{13} \rfloor}{\lfloor \frac{20}{13} \rfloor}\binom{150 \pmod{13}}{20 \pmod{13}} \equiv \binom{11}{1}\binom{7}{7} \equiv 11 \pmod{13}$. Equivalently, we have that $\binom{150}{20} \equiv \binom{8}{1}\binom{14}{3} \equiv 5 \pmod{17}$. Using the Chinese Remainder Theorem, we have $\binom{150}{20} \equiv 141 \pmod{220}$

3. Let $\angle BAE = \alpha$. From the problem, we also have $\angle EAC = \angle DBF = \angle FBC = \alpha$. Let $\angle BCA = \beta$. Since ABC is a right trangle, and D is the altutide, by similarity, we have $\angle DBA = \beta$. $\angle BFA$ can be computed by $180 - \angle FBA - \angle BAF = 180 - (\alpha + \beta) - 2\alpha$, and using $2\alpha + \beta = 90$, we have $\angle BFA = \alpha + \beta = \angle FBA$. Thus, BAF is an isosceles triangle with sides BA and AF equal, and since AG is the angle bisector, G bisects BF, so $\frac{BG}{GF} = 1$

4. We want the largest integer so that $\frac{n^2-2012}{n+7}$ is also an integer. Using long division, we obtain
$$n^2 - 2012 = (n+7)(n) + (-7n - 2012) = (n+7)n + (-7n - 49) - 1963$$
Thus $\frac{n^2-2012}{n+7}$ is an integer iff $\frac{1963}{n+7}$ is also an integer. The largest n this happens for is $1963 - 7 = 1956$.

5. We use coordinates. Let the pentagon vertices have the coordinates r, $re^{i*\frac{2\pi}{5}}$,...,$re^{i*\frac{8\pi}{5}}$ where $r = \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}$. This leads to side length being 1. The largest equilateral triangle happens when one vertex of the triangle is at a vertex of the pentagon, and the sides are symmetrically around the pentagon. WLG let the vertex of the triangle be at r. Then one side length is the line $y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}r$. Meanwhile, we want to find its intersection with the edge of the pentagon, which is
$$y - r\sin 2\pi/5 = \frac{r\sin 4\pi/5 - r\sin 2\pi/5}{r\cos 4\pi/5 - r\cos 2\pi/5}(x - r\cos 2\pi/5).$$
A straightforward application of the law of sines gives us the final answer of 1/2 sec 24.

6. Consider the state where we already have k notes, and we want to know the expected number of LilacBalls we need to open to get the remaining $n-k$ notes. With probability $\frac{n-k}{n}$ we will go to the state with $k+1$ notes, and with probability $\frac{k}{n}$ we will stay in our original state. Let E_k be the expected number of balls we need to open if we already have k distinct notes. Then, that means
$$E_k = \frac{n-k}{n}E_{k+1} + \frac{k}{n}E_k + 1$$
with our base case $E_n = 0$. Simplifying the expression, we get $E_k = E_{k+1} + \frac{n}{n-k}$, so
$$E_k = \sum_{i=k}^{n} \frac{n}{n-k}$$
and
$$E_0 = n\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right) = n \cdot H_n$$
When $n = 7$, we have $nH_n = \frac{657}{80}$.

3. Round 3

1. Let $S \in T$. Then, we have a corresponding set $S' = \{x \in S : 2013 - x\}$ in T, and we note that $\frac{A(S)+A(S')}{2} = \frac{2013}{2}$. Because each such S can be matched with a unique such S', we have that $A(R) = \frac{2013}{2}$.

2. For each hour, with the exception of 8:00 to 10:00 and 2:00 to 4:00, the minute and hour hands of a clock will form a right-angle with each other twice. From 8:00 to 10:00, and again from 2:00 to 4:00, the minute and hour hands will form a right angle only 3 times. Thus, we have
$$(8 \times 4) + (3 \times 4) = 44$$
Therefore, the minute and hour hands of a clock will form right angles with each other 44 times during one day.

3. One pile will have $n = 18$ cards and the other will have $m - n = 34$ cards.
   The algorithm for solving the problem is the following:
   From the deck of m cards, Mike will take any n cards, flip them over, and set it aside as a second pile. In the n cards, there will be x face-up cards and $n - x$ face-down cards, where $0 \leq x \leq n$. This means that in the $m - n$ cards in the other pile, there will be $n - x$ face-up cards. When Mike flips the n-card pile, there will then be x face-down cards, and $n - x$ face-up cards. Therefore, both piles will have $n - x$ face-up cards.

4. Let $x^3+ax^2+bx+c = (x-(n-1))(x-n)(x-(n+1))$. Then $a = -3n, b = (n-1)n+n(n+1)+(n-1)(n+1) = 3n^2 - 1$. $a^2/(b+1) = 9n^2/3n^2 = 3$.

5. Note that the answer doesn't even depend on p. Let B_n denote the number of blue balls after n iterations and G_n denote the number of gold balls after n iterations. We want to find $E[B_n]$. Notice that we can just iterchange blue and gold balls and still get the same result, so $E[B_n] = E[G_n]$. Also notice that $B_n+G_n = n+2$, so $E[B_n+G_n] = E[B_n]+E[G_n] = 2E[B_n] = E[n+2] = n+2$, so $E[B_n] = \frac{n+2}{2} = 262801$.

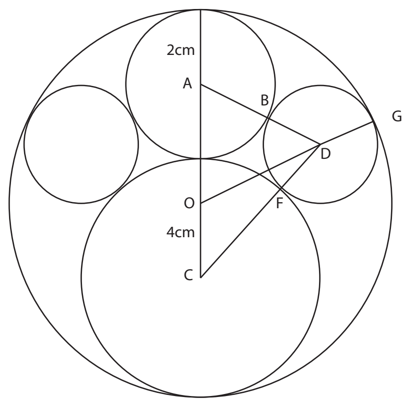
6. First, we show E lies on circle ABD, which will symmetrically show that F lies on ABD. We have $\angle ADB = \theta$, and symmetrically, $\angle BCA = \theta$. Since C and E lie on circle B, we must have $CB = EB$, so $\angle BEC = \angle BCE = \theta$. That must mean $\angle BEA = 180 - \theta$, which shows that $DAEB$ is a cyclic quadrilateral. Using this circle, we use power of a point. We have $BG \cdot AG = GF \cdot EG \to BG(BG+10) = 8(12)$. Solving this for BG gives us $BG = 6$.

4. Round 4 Part 1

1. $nS_n - (S_1 + S_2 + \ldots + S_{n-1})$

$$= n(1/1 + 1/2 + ... + 1/n) - (n - 1 * 1 + (n-2) * 1/2 + ... + 1 * 1/(n-1))$$

$$= 1 * 1/1 + 2 * 1/2 + ... + n * 1/n = n$$



2. In the diagram above, A is the center of the circle of radius 2, C is the center of the circle of radius 4, O is the center of the circle of the large circle that contains all the inside circles, D is the center of the circle whose radius we're trying to find, B is the point at which $\overline{AD}$ intersects the circle of radius 2 and the circle whose radius we're trying to find, F is the point at which $\overline{CD}$ intersects the circle of radius 4 and the circle whose radius we're trying to find, and G is the point at which the extension of $\overline{OD}$ will intersect the large circle. Letting the radius of the circle with center D be r, we can write the following:

$$\overline{AD} = 2 + r \qquad \overline{OD} = 6 - r \qquad \overline{CD} = 4 + r$$

$$\overline{AO} = 4 \qquad \overline{AC} = 8 \qquad \overline{OC} = 2$$

Now consider triangles ΔADO and ΔODC. Since the two triangles have equal altitudes and the base of ΔADO is twice the length of the base of ΔODC, the area of ΔADO is twice the area of ΔODC. By Heron's formula

$$\begin{aligned}\text{Area}(\Delta ADO) &= \sqrt{s(s-4)(s-(2+r))(s-(6-r))} \qquad \text{where} \quad s = \frac{4+(2+r)+(6-r)}{2} = 6\\ &= \sqrt{6(2)(4-r)(r)}\\ &= \sqrt{12r(4-r)}\end{aligned}$$

Similarly,

$$\begin{aligned}\text{Area}(\Delta ODC) &= \sqrt{s(s-2)(s-(4+r))(s-(6-r))} \qquad \text{where} \quad s = \frac{2+(4+r)+(6-r)}{2} = 6\\ &= \sqrt{6(4)(2-r)(r)}\\ &= \sqrt{24r(2-r)}\end{aligned}$$

Finally we have

$$2\sqrt{24r(2-r)} = \sqrt{12r(4-r)} \quad \Rightarrow \quad 4[24r(2-r)] = 12r(4-r)$$
$$\Rightarrow \quad 0 = r(7r-12)$$
$$\Rightarrow \quad r = \frac{12}{7}$$

Thus, the radius of the two identical circles that lies tangent to all three circles is $\frac{12}{7}$.

3. Notice the number of times x^k in the sum $\sum_{i=0}^{2^n-2} x^{s(i)}$ is preciesly $\binom{n}{k}$ (the number of ways to distribute the k bits among n bits). Observe this collapses to $(x+1)^n$. However, we are missing the term with all the one terms set, so the inner sum is actually equal to $(x+1)^n - x^n$. This sum telescopes to $315^{576} - 1^{576}$, so we want to take this mod 629. Notice $\varphi(629) = 576$, Both powers are equal to one, so this is equal to $\boxed{0}$

4. We count the area outside of the middle portion. Consider cutting out the middle strip. We then have the same parallelogram but with area $\frac{k-1}{k}$. Since there are two ways of cutting the middle strip out, we add this in twice to get $2\frac{k-1}{k}$. But now, we have overcounted the triangles in the corner. We can compute the area of these by slightly enlarging the opposite corners, and pasting them together to get another parallelogram. We consider enlarging two different ways. Label the topleft and bottom right areas as A, and the topright and bottom left areas as B. Then, if we scale a triangle by k, the area will scale by k^2, we have $2A(\frac{k}{k-1})^2 + 2B = \frac{k-1}{k}S$ and $2A + 2B(\frac{k}{k-1})^2 = \frac{k-1}{S}$. We want to know what $2A + 2Bis$, so adding the two equations and factoring, we have $((\frac{k}{k-1})^2 + 1)(2A + 2B) = 2\frac{k-1}{k}S$, so $2A + 2B = \frac{2(k-1)^3}{k(k^2+(k-1)^2)}S$. Thus, we have the area outside of the middle portion equal to $\frac{2(k-1)}{k} - \frac{2(k-1)^3}{k(k^2+(k-1)^2)} = \frac{2k(k-1)}{k^2+(k-1)^2}$. The area in the middle portion is $1 - \frac{2k(k-1)}{k^2+(k-1)^2} = \frac{(k-(k-1))^2}{k^2+(k-1)^2} = \frac{1}{k^2+(k-1)^2}$

5. A well known formula for Euler's totient function is $\varphi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_m})$, where $p_1, p_2, \ldots, p_m$ are the distinct prime factors of n. Thus, $\frac{n}{\varphi(n)} = \frac{\prod_{k=1}^m p_k}{\prod_{k=1}^m (p_k - 1)}$. Thus, to maximize this quantity, we want n to have as many distinct prime factors as possible (and smaller primes will contribute more to the quantity). We try $2 \cdot 3 \cdot 5 \cdot 7 = 210$, which is the best we can do. However, we can also multiply 210 by 2 to get 420, which has the same $\frac{n}{\varphi(n)}$ value, so our answer is $210 + 420 = \boxed{630}$.

6. Suppose Link always makes a right move first. We can see that a way of Link reaching the bottom right corner can be described as a set of indices where he takes turns. The set is chosen by selecting a combination of $\lfloor \frac{K}{2} \rfloor$ column indices and $\lfloor \frac{K-1}{2} \rfloor$ row indices amont $N-2$ indices in each dimension (note $\lfloor \frac{K}{2} \rfloor + \lfloor \frac{K-1}{2} \rfloor = K - 1$). Thus, the answer is $2 \cdot \binom{N-2}{\lfloor \frac{K}{2} \rfloor} \cdot \binom{N-2}{\lfloor \frac{K-1}{2} \rfloor}$. Plugging in values $N = 12, K = 6$, we have $2 \cdot \binom{10}{3} \cdot \binom{10}{2} = \boxed{10800}$

5. Round 4 Part 2

1. We note that $1 - 2 + 3 - 4 + ... + n = -(n-1)/2 + n = (n+1)/2$. Since $2012 = (n+1)/2$, then $n = 4023$. $4^{2012} \equiv 2^{4024}$, and $4023 = 3^3 * 149$. Now, we want to find 2^{4024} mod 27, and 2^{4024} mod 149. $2^{18} \cong 1$ mod 27 so $2^{4024} \cong 2^{10} \cong 25$ and $2^{148} \cong 1$ mod 149 so $2^{4024} = 2^{28} = (128)^4 = (-20)^4 = 20^4 = 123$ mod 149. Using the CRT, we get 123 mod 149 and 25 mod 27 is 3103 mod 4023.

2. Clearly $\angle PBA = 45$, so we can ignore this term for now. Consider six unit squares arranged in two rows of three. Label the top right most vertex R, the leftmost middle vertex S and the second from the right on the bottom row T. Also label the bottom left most vertex X, the bottom right most vertex Y and the topleft most vertex Z. Clearly, $\angle TRY = \angle PCA$, $\angle ZRS = \angle PDA$, and $\angle ZRY = 90$, so we know $\angle PCA + \angle PDA = 90 - \angle SRT$. Notice $\Delta STX = \Delta TRY$, and since $\angle SXT = 90$, we must have $\angle STR = 90$. In addition, $ST = RT$, so STR is an iscoceles right triangle so $\angle SRT = 45$, thus $\angle PBA + \angle PCA + \angle PDA = \boxed{90°}$

3. $\frac{9(x^2-2x+1)}{x^2-8x+16} = \frac{9(x-1)^2}{(x-4)^2}$

$$= \frac{9(x-1)^2}{(3-(x-1))^2}$$
$$= \left(\frac{3(x-1)}{3-(x-1)}\right)^2$$
$$= \left(\frac{3(x-1)}{3-(x-1)}\right)^2$$
$$= \left(\frac{3}{\frac{3}{x-1}-1}\right)^2$$

$= f(f(x))^2$

4. Let M be the tangent point of F on BC and K be the tangency point of F on BD.
We have $\angle DBC = \angle BAD = \alpha$, and $\angle DBF = \angle EAD = \frac{\alpha}{2}$ and $\angle ADE = \angle BDF = 45$, so ΔAED is similar to ΔBFD. By similarity, $\frac{AD}{BD} = \frac{ED}{DF}$, which also shows that ΔABD is similar to ΔEFD. We have $\angle BFE = 180 - \frac{\alpha}{2} - \beta - 45 = \frac{\alpha}{2} + \angle BGF$, so $\angle BGF = 45$. Now, we have $\angle FKB = \angle FMB = 90$, which implies $MG == KD$ and $BG = BD$. Symmetrically, this shows $BH = BG = BD$. Thus, since this is a simple isosceles right trangle, the area is equal to $\frac{BD^2}{2}$. In an 8,15,17 triangle, we have

$$\frac{a^2b^2}{2(a^2+b^2)} = 7200/289$$

5. Look at any sequence of k elements. The probability that it is a k-inversion is $\frac{1}{k!}$. Using linearity of expectation, the number of k-inversions is just $\binom{n}{k}\frac{1}{k!} = 7/24$.

6. Initially, we start at a score of zero. Let's look at what happens when we add the ith coin, which can be placed randomly in the range $[1, i]$. We want to look at the difference in the expected score. Suppose the coin is placed on a square with x coins. The difference in score will just be $(x+1)^2 - x^2 = 2x+1$, so all we need to do is calculate the expected value of x. But this is really simple, since we have placed $i-1$ coins and we have i squares, so this is just $\frac{i-1}{i}$. That means our answer is

$$\sum_{i=1}^{n}\left(2\cdot\frac{i-1}{i}+1\right) = 2(n-H_n)+n = 3n-2H_n = 1343/80$$

6. Championship Round

1. $2n+1 = 144169^2 = (144168+1)^2 = 4\cdot 72084^2 + 4\cdot 72084 + 1$. Thus $n+1 = 2\cdot 72084^2 + 2\cdot 72084 + 1 = 72084^2 + (72084+1)^2$ and the numbers we want are 72084 and 72085.

2. Let E_n be the maximum expected value of John's winnings if he currently has an n-sided dice. Clearly, for our base case, we can conclude $E_2 = \frac{3}{2}$.
Now, John will only want to roll an $n-1$ sided dice if the expected winnings of rolling it is higher than what he currently rolled. For example, for $n = 3$, there are three possibilities, he rolls a $1, 2$ or 3. If he rolls a 1, he will choose to roll again, but if he gets a 2 or 3, he will decide to stop since his expected winnings of continueing is less than his current total. Therefore, $E_3 = \frac{2+3}{3} + \frac{E_2}{3} = \frac{13}{6}$.
Continuing on a similar pattern, we find $E_4 = \frac{4+3}{4} + \frac{2\cdot E_3}{4} = \frac{17}{6}$, $E_5 = \frac{5+4+3}{5} + \frac{2\cdot E_4}{5} = \frac{53}{15}$, so

$$E_6 = \frac{6+5+4}{6} + \frac{3\cdot E_5}{6} = \frac{64}{15}$$

3. $f-7$ has 6 zeros at 1,2,3,4,5,6. Thus, $f-7 = (x-1)...(x-6)$. Which gives $a = -21$.

4. We note that the original condition is equivalent to saying $5a+3b$ and $5b-3a$ are both powers of 2. Without loss of generality, we may assume not both of a and b are even, otherwise $a/2$ and $b/2$ would do the job just as well. Since $5a+3b$ is a power of 2 and thus even, this means both a and b are odd. Call $5a+3b = 2^m$ and $5b-3a = 2^n$. Then $8b+2a = 2^m+2^n$. Since a and b are odd, 2^m+2^n thus has only 1 factor of 2. This can only happen if one of m or n is equal to 1. Since a and b are positive, then, $n = 1$ and $5b-3a = 2$. Meanwhile $4b+a = 2^{m-1}+1$. We notice that since $5b-3a = 2$, $b = 1$

mod 3. Since b is odd, then, let $b = 6c + 1$. It's clear that then $a = 10c + 1$ to have $5b - 3a = 2$. Then $5(10c + 1) + 3(6c + 1) = 2^m$ and thus $68c + 8 = 2^p$, so $17c + 2 = 2^{p-2}$. This means $2^{p-2} \equiv 2 \mod 17$. The smallest p this happens for is $p = 11$; thus, $17c + 2 = 512$ and $c = 30$. This gives

$$a = 301, b = 181, a + b = 482$$

satisfying all our initial conditions.

5. Draw a circle with points A, C, E. Notice D must be the center of the circle since it is equidistant from all three points. Also note $\angle AEC$ intercepts the arc formed by $\angle CDA$, and since D is in the center and E is on the circle $\angle AEC$ must be half of $\angle CDA$. But $\angle CDA$ is just 90 by definition of a square, so $\angle AEC = \boxed{45^\circ}$.

6. George will win in at most 7 moves.

Assuming that Ted's initial configuration of the coins weren't all face-up or face-down, this implies that the initial configuration must be one of the following cases: 1 face-up, 3 face-down; 2 face-up, 2 face-down; 3 face-up, 1 face-up. Since our goal is to turn all of the coins either face-up or face-down, we will regard the first and last case listed as the same case; hence, there are only two configurations we must examine.

Let's also examine the number of different moves we can make. There are actually only three different types of moves we can choose from: flipping two coins on either diagonal, flipping two coins on any one side, or flipping one coin. Note that flipping two coins on one of the diagonals is equivalent to flipping the two coins on the other diagonal; flipping two coins on any one side is equivalent to flipping the two coins on the other side (and remember that since Ted has the opportunity to rotate the index card after each of George's turn, we don't need to be concerned with which side of the index card from which we chose are two coins); lastly, flipping one coin is equivalent to flipping the other three.

Now, let's consider all of the cases along with the least number of moves that we must make in order to win. Let a 2×2 matrix represent the layout of the coins on the index card, where a u represents the coin is face-up while a d represents the coin is face-down.

Case 1:

$$\begin{bmatrix} u & d \\ d & u \end{bmatrix}$$

Notice that any rotation of this matrix will yield a matrix of the same layout. In this case, we will win if we flip either of the two diagonals. Therefore, only 1 move is required.

Case 2:

$$\begin{bmatrix} u & u \\ d & d \end{bmatrix}$$

Notice that any rotation of this matrix will yield a matrix of a similar layout (2 face-ups in a row/column and 2 face-down in the other row/column). In order to solve this case, we would hope to flip the correct row/column, which means we would only need 1 move. However, if the face-up/face-down coins are organized in rows (like above) and we flip a column or vice versa, we will get some rotation of Case 1, which we can solve in 1 move. Therefore, for Case 2, we need at most 2 moves to win.

Case 3:

$$\begin{bmatrix} u & d \\ d & d \end{bmatrix}$$

Notice that any rotation of this matrix will yield a matrix of a similar layout (1 face-up and 3 face-downs). Also, notice that

$$\begin{bmatrix} d & u \\ u & u \end{bmatrix}$$

and any rotations of this matrix will yield similar cases of the first matrix. In order to solve this case, we would hope to flip the correct 1 coin (or similarly, correct 3 coins). If we are lucky and we guessed right, then we would only need 1 move. However, if we guess wrong, we will end up in either Case 1 or Case 2, depending on which coin, or which 3 coins, we chose to flip. (I'm leaving out the number of moves needed to solve Case 3 because it includes an extra case that will be examined below. Just keep in mind that if we flip the wrong 1 coin, we will end up in either Case 1 or Case 2.)

Putting it all together:

Based on the results above, we'd want to check to see whether or not the initial configuration falls under Case 1, which means we only need 1 move to solve. Thus, our first move should be to flip the 2 coins on either diagonal. If Case 1 was indeed the initial configuration, then we're done. However, if not, then the initial configuration falls under either Case 2 or Case 3. Note that in either of the two cases, flipping the 2 coin on either diagonal will actually keep the configuration of the coins in the same case. (Flipping a diagonal in Case 2 will yield a configuration in Case 2, and the same goes for Case 3.) Next, we'd want to check to see which of the 2 remaining cases the coins are now in. Based on the results above, we'd want to check to see if the configuration of the coins fall under Case 2, which means we only need 2 moves to win. If Case 2 was indeed the initial configuration, then we would have taken a total of 3 moves to win. If not, then that means the initial configuration was in Case 3. (Note that the moves we have made so far, flipping 2 of a diagonal and flipping 2 in any row/column, will send a Case 3 configuration to another Case 3 configuration. This means if we still haven't won by move 3, the initial configuration must have been a Case 3.) Thus, for our next, 4th move, we will want to flip any 1 coin (similarly, any 3 coins). If we guessed correctly, then will have won in 4 moves. However, if we guessed wrong, then our configuration is now a Case 1 or Case 2. This then brings us back to where we started. Between Case 1 and Case 2, we want to check for a Case 1 first by flipping any diagonal. If it was a Case 1, then we would have taken 5 moves to win. If not, then then it must have been a Case 2 which requires 2 moves to win, bringing us to a total of a maximum of 7 moves to win.

7. Consolation Round

1. This is caused by having exactly two increasing sequences of numbers with a descent in the middle. For each one of these, we can consider the set A consisting of elements in the first sequence, and B consisting of elements in the second sequence. There are 2^n ways of creating two sets like that. However, $n+1$ of those ways won't work, as then A's elements arranged in increasing order, concatenated with B's elements arranged in increasing order, will result in the sequence $1, 2, ..., n$ which has no descents. Thus the total number of ways to have exaclty one descent is $2^n - n - 1$.

2. WLOG let each triangle have sides a, b, c where $a < b < c$. Of course $a + b + c = 2012$
   For a fixed c, how many different triangles can we make with perimeter 2012.
   First of all, see that $\left\lceil \frac{2012}{3} \right\rceil = 671 < c < \left\lceil \frac{2012}{2} - 1 \right\rceil = 1005$
   Now since $b \le c$ and $b = 2012 - a - c$, we have $2012 - a - c \le c$ or rather $a \ge 2012 - 2c$
   And also since $a \le b = 2012 - a - c$, $2a \le 2012 - c$
   Now suppose $c \in [671, 1005]$ and $a \in \left[2012 - 2c, \left\lfloor \frac{2012-c}{2} \right\rfloor\right]$ and let $b = 2012 - a - c$ (the above calculations show that no other positive choices of a, b, and c add to 2012 and satisfy the triangle inequality)
   Verify that the triangle inequality is satisfied for any such choice of a and c
   Now we see that if c is even, there are $\frac{2012-c}{2} - (2012 - 2c) + 1 = \frac{3c}{2} - 1005$ possible triangles we can make and if c is odd there are $\frac{2012-c-1}{2} - (2012 - 2c) + 1 = \frac{3c-1}{2} - 1005$ possible triangles.
   So for the even $c's$, the sequence of numbers of possible triangles goes:

672	674	676	...	1004
3	6	9	...	501

   and the sequence has 167 terms so the sum is $\frac{(3+501)*167}{2} = 252 * 167$
   And for the odd $c's$, the sequence is

671	673	675	...	1005
1	4	7	...	502

   and the sequence has 168 terms so the sum is $\frac{(1+502)*168}{2} = 503 * 84$ so the total number of such triangles is $252 * 167 + 503 * 84$.

3. A well known formula for Euler's totient function is $\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_m})$, where $p_1, p_2, \ldots, p_m$ are the distinct prime factors of n. Thus, $\frac{n}{\varphi(n)} = \frac{\prod_{k=1}^{m} p_k}{\prod_{k=1}^{m}(p_k - 1)}$. Notice the only even prime is 2, so 2 must be a factor of n, and there can only be exactly one other prime factor of n. We plug in three and notice the quantity is equal to three, so n can only have primes $2, 3$. No other primes will work. So we have changed this to the equivalent problem of calculating the sum of the recipricols of the numbers whose primes factors are exactly $2, 3$. But this is not too difficult to calculate. Consider the sum $\left(\sum_{i=1}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=1}^{\infty} \frac{1}{3^j}\right)$. We claim this product holds all our desired numbers. To see why this is true, every

term in the sum will be in the form $\frac{1}{2^i 3^j}$, for any i, j, so this satisfies all our constraints. The sums can be calculated by sum of an infinite geometric series, giving us $1 \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$.

4. We claim that $f(1) + ... + f(2^n) = \frac{2^{2n}-1}{3} + 1$, and we use induction to prove this. When $n = 1$, we have $f(1) + f(2) = 2$, satisfying our base case. Suppose it's true for 2^{n-1}. Then for 2^n,

$$\begin{aligned} f(1) + ... + f(2^n) &= f(1) + f(3) + ... + f(2^n - 1) + f(2) + ... + f(2^n) \\ &= 1 + 3 + ... + 2^n - 1 + f(1) + ... + f(2^{n-1}) \\ &= 2^n \cdot 2^{n-2} + \frac{2^{2(n-1)} - 1}{3} + 1 \\ &= \frac{4 \cdot 2^{2(n-1)} - 1}{3} + 1 \\ &= \frac{2^{2n} - 1}{3} + 1, \end{aligned}$$

completing the induction. Then,

$$f(1) + ... + f(30) = f(1) + ... + f(32) - 31 - 1 = 1023/3 - 31 = 310$$

5. Consider adding one square to the triangle. By similarity, we must have $\frac{b}{x} = \frac{h}{h-x}$, so $x = \frac{bh}{b+h}$. Let $h_0 = h, b_0 = b$, and $b_n = \frac{b_{n-1}h_{n-1}}{b_{n-1}+h_{n-1}}$, and $h_n = h_{n-1} - b_n$. By induction, we can prove that $b_n = b\left(\frac{h}{b+h}\right)^n$, and $h_n = h\left(\frac{h}{b+h}\right)^n$. We sum this infinite series of b_n^2, to get $\frac{bh^2}{b+2h} = 1090000/209$

6. Let D be the directrix of the parabola. Let D_1, D_2 be the point on the directrix closest to P_1, P_2 respectively. Let $\angle D_1P_1F = 2\alpha, \angle D_2P_2F = 2\beta$. Consider the two lines l_1, l_2 going through points D_1, P_1 for l_1 and D_2, P_2 for l_2. Then, since l_1, l_2 are parallel, we have that $2\beta + 2\alpha = 180$. Thus, $\beta + \alpha = 90$, so $\angle P_1QP_2 = 90$