Time Limit: 60 mins.
Maximum Score: 125 points.
Instructions:

1. When a problem asks you to "compute" or "list" something, no proof is necessary. However, for all other problems, unless otherwise indicated, you must justify your answers.

2. You may freely assume results of a previous problem in proving later problems, even if you have not proved the previous result.

3. You may use both sides of the paper and multiple sheets of paper for a problem, but separate problems should be on separate sheets of paper. Label the pages of each problem as 1/2, 2/2, etc., in the upper right hand corner. Write your team ID at the upper-right corner of every page you turn in.

4. Partial credit may be given for partial progress on a problem, provided the progress is sufficiently nontrivial.

5. Throughout this round, unless stated otherwise, all groups are assumed to be finite, meaning they contain finitely many elements.

6. When you see a product of two group elements which represent permutations, symmetry operations, or group actions in general, for example gh, you should read it from **right to left**: first apply h, and then apply g. This is the same convention as for function composition.

7. *You may not use without proof results that are not discussed in this round.*

8. **Calculators are not allowed!**

Introduction: This power round deals with symmetry groups. We will begin by introducing groups, which are just mathematical objects that satisfy some rules. However, even though this definition is abstract, we will discuss a more intuitive view of groups as the set of symmetries of certain objects. The Zome kits provided may be helpful in visualizing the polyhedra and their symmetries. The general aim of this round is to provide a more visual and hands-on view of a subject that is often perceived as dry and abstract, and to give you a taste of the power of group theory. One of the things that group theory can show us is the equivalence between seemingly different types of objects, such as symmetries of a polyhedron and permutations of a set. Throughout this round, you will encounter many such equivalences.

Although different team members may work on different rounds, it is recommended that all team members at least read the first section, "Introduction to Groups," so that they are aware of the relevant definitions. Many of the sections can be solved independently, and those that require definitions and results from previous sections will say so in their introductions.

Introduction to Groups (20 Points)

We will begin with the definition of a group. Though it may seem a bit abstract, once we start discussing groups as symmetries of objects, the concepts should hopefully become more intuitive.

A **group**, $(G, \cdot)$ is a set, G, with a **binary operation** $\cdot$, that takes as inputs two elements of the set G and outputs another element in the set. I.e., if g and h are elements of the set G, then $g \cdot h$ is also in the set G. In addition, a group must satisfy the following properties:

a. **Associativity**: If g, h, and k are in G, then $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

b. **Identity Element**: There exists an element 1 in G such that, for every element g in the group, $g \cdot 1$ $= 1 \cdot g = g$. Note that this "1" is not the same as the number 1, but is a notation used to denote the identity element. However, since our groups will be written multiplicatively, and the number 1 is a multiplicative identity, this notation is a convenient way of representing the identity.

c. **Inverse Element**: For every element g in G, there exists an inverse g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$, where 1 is the identity element. Note that both directions must equal 1 for g^{-1} to actually be an inverse.

For convenience, we will refer to a group as G instead of $(G, \cdot)$. An **element** of a group is just an element of its underlying set. In addition, we will usually omit the $\cdot$ when writing the binary operation: i.e., $g \cdot h$ will just be written as gh. We will also generally refer to the binary operation as multiplication.

A group G is said to be **abelian** if multiplication is commutative: if for all elements g, h in G, $gh = hg$. Note that groups are not abelian in general.

1. [**2**] Prove that the integers, $\mathbb{Z}$, form an abelian group under addition.

Note that the integers are not a group under multiplication, since no integer besides 1 and -1 have a multiplicative inverse: 1 is clearly the identity, but there is no integer n such that $5n = 1$, so 5 is not invertible, and so $\mathbb{Z}$ is not a group under multiplication.

2. State why the following are not examples of groups:

    a. [**2**] The nonnegative integers, $\mathbb{N}_0$, under addition.

    b. [**2**] The rational numbers, $\mathbb{Q}$, under multiplication.

3.   a. [**2**] Prove that the identity of a group is unique, and that every element has a unique inverse. In other words, show that if 1 and $1'$ are identities, then $1 = 1'$, and that if a and b are inverses of g, then $a = b$.

    b. [**2**] Prove that if g and h are elements of a group G, then $(gh)^{-1} = h^{-1}g^{-1}$.

    c. [**2**] Prove the cancellation laws: If g, h, and k are elements of a group, and if $gh = gk$, then $h = k$; Similarly, if $gk = hk$, then $g = h$. Conclude that if $gh = g$ or $hg = g$, then $h = 1$.

4. [**3**] For an element g in G, g^k, where k is a positive integer, is the result of multiplying g by itself k times. In addition, $g^0 = 1$ and $g^{-k} = (g^{-1})^k$. The **order of a group element** g is the smallest positive integer k such that $g^k = 1$. If there is no such positive integer, we say the order of g is ∞. For example, 1 has order 1. Also, the **order of a group** is the number of elements in it. Prove that, if G has finite order, so does any element g in G.

5. [**2**] If g is an element of order n, prove that $g^{-1} = g^{n-1}$.

6. [**3**] Prove that, if g is an element of a group G and $g^k = 1$, where k is a positive integer, then k is a multiple of the order of g.

Special Types of Groups

In the following sections, we will familiarize ourselves with some special types of groups and begin to see how they can be used to describe the symmetries of polyhedra.

Cyclic Groups (15 Points)

We will begin by introducing the simplest type of group: the **cyclic** groups. If C is a cyclic group, then every element c in C can be written as $c = g^k$, for some g in G. Such an element g is called a **generator** of the cyclic group (we will define generators for other groups later).

7. A **symmetry operation** on an object is a rotation, reflection, or possibly a combination of both, that keeps the object looking the same as it was originally. These operations form a group under composition, which is just applying one symmetry operation after another: If σ and τ are symmetry operations, then $\sigma\tau$ is a symmetry operation that represents applying τ and then σ. Note that we always read composed operations from right to left. The identity is just the operation of doing nothing, and the inverse of a symmetry operation is just the operation that reverses it, returning to the original position. Throughout this round, we will mostly be only considering the rotations, so you should only consider rotations unless told otherwise.

   a. [**2**] Consider a regular pentagonal pyramid, which you can build one with the provided Zome kits, using just blue struts. Describe the elements of the (rotational) symmetry group of the pentagonal pyramid. This will be the group of rotations of the pyramid that leaves it indistinguishable from its original state.

   b. [**2**] Show that this rotation group is cyclic, meaning that any rotation can be achieved by just applying some rotation some number of times.

   c. [**2**] Show that this rotation group is equivalent to the group of integers mod 5 under addition, in the sense that each rotation can be associated to an integer mod 5 such that applying one rotation after another is associated with the sum of the associated integers of the two rotations.

8. Let C_n be the cyclic group generated by an element, g, of order n.

   a. [**3**] Show that C_n's elements are exactly $\{1, g, g^2, \ldots, g^{n-1}\}$.

   b. [**3**] What is the order of g^k, in terms of k and n? Given this, what must be true of k and n for g^k to be a(nother) generator?

   c. [**3**] Prove that all cyclic groups are abelian.

It turns out that any finite cyclic group C_n is equivalent to the rotational symmetry group of a regular $n$gonal pyramid, and also to the group of integers mod n under addition.

Dihedral Groups (10 Points)

We will now discuss a new type of group, called the **dihedral** groups. D_{2n}, the dihedral group of order $2n$, is the symmetry group of a regular $n$gon, including both rotations and reflections (this is one case where we will consider reflections). We can see that there are n ways to rotate the $n$gon (including the identity operation), and that there are n reflection axes, giving a total of $2n$ symmetry operations as elements of D_{2n}. Let r be the element of D_{2n} that represents a clockwise rotation by $\dfrac{360}{n}$ degrees, and let s represent reflection over a fixed axis. It is not difficult to see that r has order n and s has order 2, and that r and s generate the entire group.

9. [**2**] Consider an equilateral triangle with vertices labeled 1, 2, and 3 in clockwise order, with 1 pointing up, and let s represent reflection about the vertical axis. Draw the triangle after the symmetry operation sr is performed, i.e., after first applying r and then applying s (by convention, operations are applied from right to left, similar to function composition). It may help to build a triangle from the blue Zome struts to help you visualize the symmetry group.

10. a. [**5**] Prove that sr^k has order 2 for all $0 \leq k \leq n-1$, and conclude that $sr^k = r^{-k}s$.

    b. [**3**] Show that D_{2n} is not abelian when $n > 2$.

Note that, though D_{2n} can be seen as the full symmetry group of a regular $n$gon, it can also be seen as the rotation group of an $n$gonal prism. Thus, the cyclic and dihedral groups, the simplest groups, can be thought of as the rotation groups of pyramids and prisms, the simplest polyhedra. To describe the rotation groups of more complicated polyhedra, we will need to introduce two new types of groups.

Symmetric Groups (20 Points)

S_n, the **symmetric** group on n letters, is the group of permutations of n objects. Formally, a **permutation** is a function from from the set of first n positive integers to itself such that the function does not send any two integers to the same integer. Intuitively, you can think of a permutation as a "reordering" of these n integers. Since permutations can be thought of as functions, the group operation will be function composition, i.e., applying one permutation after another. Thus, S_n has order $n!$, which is the number of ways to permute the n integers.

We will now introduce a very useful notation, known as cyclic notation, for representing elements of S_n. In this notation, elements are written as products of disjoint cycles, where each number in a cycle is sent to the next number, with the last number being sent to the first. For example, if we consider S_3 acting on the ordered set $\{1, 2, 3\}$, the element of S_3 that sends 1 to 2, 2 to 1, and 3 to 3, resulting in $\{2, 1, 3\}$, can be written in this notation as $(12)(3)$. Cycles of length 1 can be omitted, giving the more concise (12). In a similar fashion, (123) represents sending 1 to 2, 2 to 3, and 3 to 1, resulting in $\{2, 3, 1\}$. The identity permutation is still written as 1. Note that $(12) = (21)$ and $(123) = (231) = (312)$. One can see after some thought that this cycle decomposition is unique up to the ordering of the cycles and circular permutations of the numbers in each cycle. By convention, we write the smallest number in the cycle first, and we will order the cycles by the first number in each. From now on, all elements of symmetric groups will be written in cyclic notation.

11. [**2**] Compute the inverses of the following permutations: (1 2), (1 2 3 4 5), and (1 2)(3 4 5)(6 8 7 9). No proof is required.

12. [**2**] Compute the following product of permutations: (1 2 3)(2 3 4)(2 1 4). Recall that permutations are applied from the right to the left. No proof is required.

13.   a. [**2**] Using cyclic notation, write down all the elements of S_3.

        b. [**3**] Prove that S_3 is equivalent to D_6, the symmetry group of an equilateral triangle.

        c. [**3**] Prove that S_n is not abelian for $n > 2$.

14. [**3**] Show that, when written as the product of disjoint cycles, the order of an element of a symmetric group is the least common multiple of the lengths of all of its cycles. You can assume that disjoint cycles commute, i.e., if σ and τ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

15. [**5**] Prove that S_n is generated by all of its 2-cycles, i.e., its transpositions. This means that you can get any permutation in S_n by composing some number of transpositions.

Alternating Groups (20 Points)

Note: It is strongly recommended that you at least read the section on symmetric groups before doing these problems.

A_n, the **alternating** group on n letters, is the group of even permutations of S_n. Recall that any element of S_n can be written as a product of transpositions. A_n then consists of those elements of S_n that can be written as a product of an even number of transpositions. Such elements are called **even permutations**, while elements of S_n that can be written as a product of an odd number of transpositions are called **odd permutations**.

16.   a. [**8**] Prove that a permutation σ in S_n cannot be both even and odd: that is, show that σ cannot be written both as a product of an even number of transpositions and a product of an odd number of transpositions. Possible hint: begin by proving that 1 is even and not odd.

        b. [**4**] Prove that the product of two even and of two odd permutations is even, while the product of an even and an odd permutation is odd. Conclude that the order of A_n is $\dfrac{n!}{2}$ for $n > 1$.

        c. [**3**] Show that, when written in cyclic notation, the elements of A_n are those elements of S_n with an even number of cycles of even length. Use this fact to list the elements of A_4.

17. [**5**] Prove that A_n is generated by all of its 3-cycles. This means that any permutation in A_n can be achieved by just composing some number of 3-cycles.

Although Symmetric and Alternating groups may currently seem unrelated to symmetry groups, we will see in the next section that the rotation groups of the regular polyhedra are just these types of groups.

Regular Polyhedra and Their Rotation Groups (40 Points)

Note: It is strongly recommended that you at least read the sections on symmetric and alternating groups before doing the problems in the rest of this round.

The regular polyhedra, also known as the Platonic Solids, are polyhedra whose faces are all identical regular polygons, and whose vertices are all identical, having the same number of these polygons meeting at each vertex. The 5 Platonic Solids are listed below. These are the only five Platonic Solids, though we will not prove this here.

The **tetrahedron** has 3 triangles around each vertex. It has 4 faces, 4 vertices, and 6 edges.

The **cube** has 3 squares around each vertex. It has 6 faces, 8 vertices, and 12 edges.

The **octahedron** has 4 triangles around each vertex. It has 8 faces, 6 vertices, and 12 edges.

The **dodecahedron** has 3 pentagons around each vertex. It has 12 faces, 20 vertices, and 30 edges.

The **icosahedron** has 5 triangles around each vertex. It has 20 faces, 12 vertices, and 30 edges.

It is highly encouraged that you construct all five of them with the provided Zome kits. The tetrahedron and octahedron can be constructed with just green struts, and the other three can be constructed with just blue struts.

Every polyhedron has a **dual** polyhedron, which can be constructed by placing a point at the center of each of the original polyhedron's faces, and connecting each resulting point with the resulting points of all neighboring faces of the original polyhedron. It is straightforward to see that the tetrahedron is self-dual, that the cube and octahedron are dual to each other, and that the icosahedron and dodecahedron are dual to each other.

18. [**5**] Prove that the rotation group of a tetrahedron is equivalent to A_4 by considering its effect on the vertices of the tetrahedron, and by computing its order.

Group Actions

A group G is said to **act** on a set S if, for each element g of G, g acts like a function from S to S, sending elements of S to elements of S. The only restrictions are that, if s is an element of S, $1(s) = s$ (the identity sends s to itself) and if g and h are elements of G, $h(g(s)) = (hg)(s)$, which means that applying g and then h to s gives the same result as applying hg to s. From now on, we will write $g(s)$ as $g \cdot s$ for brevity.

Examples of group actions are S_n acting on the set $\{1, \ldots, n\}$ by permutation, or D_{2n} acting on the vertices of an $n$gon by symmetry operations. We will soon study the action on the Platonic Solids by their rotation groups.

The **orbit** of an element s of S under a group G, denoted O_s. is the set of all elements of S that s can be sent to by elements of G. In other words, $O_s = \{g \cdot s \mid g \in G\}$.

The **stabilizer** of s, denoted as G_s is the set of all elements of G that send s to itself. In other words, $G_s = \{g \in G \mid g \cdot s = s\}$.

19. [**2**] Show that, if G acts on S, and a and b are elements of S such that $b = g \cdot a$ for some g in G, then $a = g^{-1} \cdot b$.

20. a. [**3**] Find the number of elements in the orbit and stabilizer of the center of one of a cube's faces under its rotation group. Here, the rotation group of the cube acts on the face centers.

    b. [**3**] Repeat part a, using the midpoint of an octahedron's edge.

    c. [**8**] Orbit-Stabilizer Theorem: Prove that if G acts on S, and s is an element of S, then
    $$|G| = |O_s| \cdot |G_s|.$$

    d. [**6**] Use the Orbit-Stabilizer Theorem and your answers to parts a and b to find the order of the cubic and octahedral rotation groups. In addition, prove that both of these rotation groups are equivalent to S_4 by finding a set of 4 objects in the cube and octahedron that can be completely permuted by the respective rotation groups. (You needed to compute the order of the rotation group first to make sure that it wasn't a larger group that just contained S_4 as a subgroup).

You have shown that the cubic and octahedral rotation groups are equivalent. This turns out to be a general property of dual polyhedra, which means that the dodecahedral and icosahedral rotation groups are also equivalent. This is because the vertices of one polyhedron correspond to the faces of its dual and vice versa, while the edges in one polyhedron correspond to the edges in its dual.

21. [**3**] Use the Orbit-Stabilizer Theorem to determine the order of the icosahedral/dodecahedral rotation group.

22. [**10**] Consider a dodecahedron. As shown in the figure below, you can inscribe a cube in the dodechaedron by drawing some of the diagonals of the dodecahedron's pentagonal faces. By drawing all 5 diagonals of all 12 pentagonal faces, 5 cubes can be inscribed in a dodecahedron. Show that there is some rotation that achieves each even permutation of these 5 cubes and, combined with your knowledge of the order of this group, conclude that the dodecahedral, and thus the icosahedral, rotation group is equivalent to A_5.
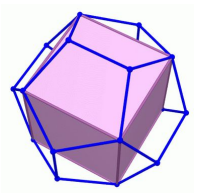
Conclusion (0 Points)

We have now seen polyhedra whose rotation groups are the cyclic groups, the dihedral groups, and the Platonic groups (tetrahedral, octahedral, icosahedral). It turns out that these are the only finite rotation groups in 3 dimensions. We have actually developed all of the tools needed to prove this result, so if anyone is interested in the proof, it will be provided in the solutions. Feel free to try to prove it yourself, though no points will be given for it.