Time limit: 60 minutes.
Instructions: This test contains 10 short answer questions. All answers are positive integers. Only submitted answers will be considered for grading.
No calculators.

1. How many permutations of the set $\{B, M, T, 2, 0\}$ do not have B as their first element?

2. Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

3. Compute the remainder when 98! is divided by 101.

4. Three lights are placed horizontally on a line on the ceiling. All the lights are initially off. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

5. Let P be the probability that the product of 2020 real numbers chosen independently and uniformly at random from the interval $[-1, 2]$ is positive. The value of $2P - 1$ can be written in the form $(\frac{m}{n})^b$, where m, n and b are positive integers such that m and n are relatively prime and b is as large as possible. Compute $m + n + b$.

6. Let N be the number of non-empty subsets T of $S = \{1, 2, 3, 4, \ldots, 2020\}$ satisfying $\max(T) > 1000$. Compute the largest integer k such that 3^k divides N.

7. Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ac = abc + 1$.

8. Dexter is running a pyramid scheme. In Dexter's scheme, he hires *ambassadors* for his company, Lie Ultimate. Any ambassador for his company can recruit up to two more ambassadors (who are not already ambassadors), who can in turn recruit up to two more ambassadors each, and so on (Dexter is a special ambassador that can recruit as many ambassadors as he would like). An ambassador's *downline* consists of the people they recruited directly as well as the downlines of those people. An ambassador earns *executive status* if they recruit two new people and each of those people has at least 70 people in their downline (Dexter is *not* considered an executive). If there are 2020 ambassadors (including Dexter) at Lie Ultimate, what is the maximum number of ambassadors with executive status?

9. For any point (x, y) with $0 \leq x < 1$ and $0 \leq y < 1$, Jenny can perform a *shuffle* on that point, which takes the point to $(\{3x + y\}, \{x + 2y\})$ where $\{\alpha\}$ denotes the fractional or decimal part of α (so for example, $\{\pi\} = \pi - 3 = 0.1415...$). How many points p are there such that after 3 *shuffles* on p, p ends up in its original position?

10. Let $\psi(n)$ be the number of integers $0 \leq r < n$ such that there exists an integer x that satisfies $x^2 + x \equiv r \pmod{n}$. Find the sum of all distinct prime factors of

$$\sum_{i=0}^{4} \sum_{j=0}^{4} \psi(3^i 5^j).$$