1. For lunch, Lamy, Botan, Nene, and Polka each choose one of three options: a hot dog, a slice of pizza, or a hamburger. Lamy and Botan choose different items, and Nene and Polka choose the same item. In how many ways could they choose their items?

   Answer: 18

   Solution: There are 3 ways for Lamy to choose an item. After that, there are 2 ways for Botan to choose a different item from Lamy. Then there are 3 ways for Nene to choose an item, and after that there is just 1 way for Polka to choose the same item as Nene. The number of ways for them to choose the items is $3 \cdot 2 \cdot 3 \cdot 1 = \boxed{18}$.

2. Compute the number of positive integer divisors of 100000 which do not contain the digit 0.

   Answer: 11

   Solution: Note that $100000 = 2^5 \cdot 5^5$. Any multiple of 10 ends in a 0, so a divisor of $2^5 \cdot 5^5$ that does not contain a 0 is either not divisible by 2 or not divisible by 5. We consider the cases separately.

   - A divisor of $2^5 \cdot 5^5$ that is not divisible by 5 will not contain the prime factor 5, and thus it is either 1 or only contains the prime factor 2, so it must be a power of 2. The powers of 2 up to 2^5 are 1, 2, 4, 8, 16, and 32.
   - A divisor of $2^5 \cdot 5^5$ that is not divisible by 2 will not contain the prime factor 2, so it is either 1 or only contains the prime factor 5, and thus it must be a power of 5. The powers of 5 up to 5^5 are 1, 5, 25, 125, 625, and 3125.

   Since 1 appears in both lists, the total number of divisors of $2^5 \cdot 5^5$ that do not contain a 0 is $6 + 6 - 1 = \boxed{11}$.

3. Katie and Allie are playing a game. Katie rolls two fair six-sided dice and Allie flips two fair two-sided coins. Katie's score is equal to the sum of the numbers on the top of the dice. Allie's score is the product of the values of two coins, where heads is worth 4 and tails is worth 2. What is the probability Katie's score is strictly greater than Allie's?

   Answer: $\frac{25}{72}$

   Solution: Allie's score is 16 with probability $\frac{1}{4}$, 8 with probability $\frac{1}{2}$, or 4 with probability $\frac{1}{4}$.

   - If Allie's score is 16, Katie can't win.
   - If Allie's score is 8, then there are 10 arrangements of Katie's dice in which Katie beats Allie.
   - If Allie's score is 4, there are 30 arrangements of Katie's dice in which Katie beats Allie.

   In total, the probability Katie's score is (strictly) greater than Allie's is $\frac{2 \cdot 10 + 30}{6^2 \cdot 2^2} = \boxed{\frac{25}{72}}$.

4. Richard and Shreyas are arm wrestling against each other. They will play 10 rounds, and in each round, there is exactly one winner. If the same person wins in consecutive rounds, these rounds are considered part of the same "streak". How many possible outcomes are there in which there are strictly more than 3 streaks? For example, if we denote Richard winning by R and Shreyas winning by S, $SSRSSRRRRR$ is one such outcome, with 4 streaks.

   Answer: 932

Solution: The total number of possible outcomes is $2^{10} = 1024$ because there are two possibilities of the winner for each of 10 rounds. Next, let us find the amount of outcomes with 3 or fewer streaks.

Fix some $k \in \{1, 2, 3\}$; we compute the number of outcomes with exactly k streaks. For there to be k streaks, one first has to find $k-1$ places in which two consecutive games have different winners, as these are the places where a new streak starts. This will then divide the 10 games into k streaks. There are 9 places at which this can occur, because this cannot happen before the first game or after the last game. Thus, there are $\binom{9}{k-1}$ ways to choose the places where a new streak starts. (Note this logic also works for the 1-streak case, in which the winner never changes.) Then, there are 2 choices for the winner in the first streak; this then uniquely determines the winners in the remaining games. So, there are

$$2\binom{9}{k-1}$$

total outcomes in which there are k streaks. Summing over each $k \in \{1, 2, 3\}$, there are

$$2\binom{9}{0} + 2\binom{9}{1} + 2\binom{9}{2} = 2 + 18 + 72 = 92$$

outcomes with 3 or fewer streaks. Subtracting from the total, there are $1024 - 92 = \boxed{932}$ outcomes with strictly more than 3 streaks.

5. Given a positive integer n, let $s(n)$ denote the sum of the digits of n. Compute the largest positive integer n such that $n = s(n)^2 + 2s(n) - 2$.

Answer: 397

Solution: Let d denote the number of digits in n. Note that we cannot have $d \geq 5$: because $s(n) \leq 9d$, we must have

$$10^{d-1} \leq n \leq (9d)^2 + 2 \cdot 9d - 2.$$

In particular, $10^{5-1} > (9 \cdot 5)^2 + 2 \cdot 9 \cdot 5 - 2$, with the left-hand side increasing much faster than the right-hand side, so $d \geq 5$ do not satisfy the condition.

Additionally, we cannot have $d = 4$. Because $n \leq (9 \cdot 4)^2 + 2(9 \cdot 4) - 2 < 1400$, we have $s(n) \leq 1 + 3 + 9 + 9 = 22$, so

$$n \leq 22^2 + 2 \cdot 22 - 2 < 1000.$$

Therefore, n has at most 3 digits, so $s(n) \leq 9 \cdot 3 = 27$. Now, observe that

$$n = s(n)^2 + 2s(n) - 2 \equiv n^2 - n + 1 \pmod 3,$$

so $(n-1)^2 \equiv 0 \pmod 3$, and thus $n \equiv 1 \pmod 3$. We now do casework on $s(n)$.

- If $s(n) = 25$, then $n = 25^2 + 2 \cdot 25 - 2 = 673$, contradiction.
- If $s(n) = 22$, then $n = 22^2 + 2 \cdot 22 - 2 = 526$, contradiction.
- If $s(n) = 19$, then $n = 19^2 + 2 \cdot 19 - 2 = 397$, which works.
- If $s(n) < 19$, then $n < 397$, which is less.

Thus, the answer is $\boxed{397}$.

6. Bayus has eight slips of paper, which are labeled 1, 2, 4, 8, 16, 32, 64, and 128. Uniformly at random, he draws three slips with replacement; suppose the three slips he draws are labeled a, b, and c. What is the probability that Bayus can form a quadratic polynomial with coefficients a, b, and c, in some order, with 2 distinct real roots?

   Answer: $\mathbf{\frac{111}{128}}$

   Solution: We compute the complement: namely, we compute the probability that regardless of the ordering of a, b, and c, no quadratic Bayus makes will have 2 distinct real roots. For this to be the case, it is sufficient that the largest possible discriminant is nonpositive. Without loss of generality, assume $b = \max(a, b, c)$, so that the largest possible discriminant is $b^2 - 4ac$. Now, let $x = \log_2 a$, $y = \log_2 b$, and $z = \log_2 c$, so that x, y, z are integers satisfying $y = \max(x, y, z)$. Then

$$0 \geq b^2 - 4ac = 2^{2y} - 4 \cdot 2^x \cdot 2^z,$$

   so

$$x + z + 2 \geq 2y \geq x + z.$$

   Thus, $x + z \in \{2y, 2y - 1, 2y - 2\}$, so the unordered pair $\{x, z\}$ is one of $\{y, y\}$, $\{y, y - 1\}$, $\{y, y - 2\}$, or $\{y - 1, y - 1\}$.

   Now, we lift our assumption that $y = \max(x, y, z)$ to compute the answer. We have four cases.

   - Suppose (x, y, z) is some ordering of (t, t, t). There are 8 such ordered triples.
   - Suppose (x, y, z) is some ordering of $(t, t, t - 1)$. There are $3 \cdot 7 = 21$ such ordered triples.
   - Suppose (x, y, z) is some ordering of $(t, t-1, t-1)$. There are $3 \cdot 7 = 21$ such ordered triples.
   - Suppose (x, y, z) is some ordering of $(t, t, t - 2)$. There are $3 \cdot 6 = 18$ such ordered triples.

   Subtracting, the probability is $1 - \frac{8+21+21+18}{8^3} = \boxed{\frac{111}{128}}$.

7. Luke the frog has a standard deck of 52 cards shuffled uniformly at random placed face down on a table. The deck contains four aces and four kings (no card is both an ace and a king). He now begins to flip over the cards one by one, leaving a card face up once he has flipped it over. He continues until the set of cards he has flipped over contains at least one ace and at least one king, at which point he stops. What is the expected value of the number of cards he flips over?

   Answer: $\mathbf{\frac{689}{45}}$

   Solution: Let's first compute the expected number of cards until Luke flips over an ace. Note that we can view this situation as computing the expected position of the first ace in a random permutation of the 52 cards. Each permutation is of the form

$$\_A\_A\_A\_A\_,$$

   where the As are aces and the blanks refer to the cards between the aces. For each of the $52 - 4$ cards other than the aces, it will be in each of the 5 gaps with equal probability, so there is a $\frac{1}{5}$ probability that it is before the first ace. Thus, by Linearity of Expectation, the expected number of cards up to and including the first ace is $\frac{52-4}{5} + 1 = \frac{53}{5}$. Similarly, the expected number of cards to flip over the first king is also $\frac{53}{5}$.

   We now return to the original problem. Let the random variable A be the number of cards needed to flip over until Luke sees the first ace, and define K similarly for the first king. Our answer is $\mathbb{E}[\max(A, K)]$. Now, we recall the property $\max(a, b) = a + b - \min(a, b)$, so again by Linearity of Expectation,

$$\mathbb{E}[\max(A, K)] = \mathbb{E}[A] + \mathbb{E}[K] - \mathbb{E}[\min(A, K)].$$

We have already calculated $\mathbb{E}[A] = \mathbb{E}[K] = \frac{53}{5}$. The last term is calculated analogously with 8 dividers (for kings and aces) instead of 4, giving us a value of $\frac{52-8}{9} + 1 = \frac{53}{9}$. Thus, our answer is $\frac{53}{5} + \frac{53}{5} - \frac{53}{9} = \boxed{\frac{689}{45}}$.

8. Define the two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by $a_0 = 3$ and $b_0 = 1$ with the recurrence relations $a_{n+1} = 3a_n + b_n$ and $b_{n+1} = 3b_n - a_n$ for all nonnegative integers n. Let r and s be the remainders when a_{32} and b_{32} are divided by 31, respectively. Compute $100r + s$.

Answer: 3010

Solution 1: Set $p = 31$. Define $\iota, \sigma \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ by $\iota(x, y) = (x, y)$ and $\sigma(x, y) = (y, -x)$ so that

$$(a_{n+1}, b_{n+1}) = 3(a_n, b_n) + (b_n, -a_n) = (3\iota + \sigma)(a_n, b_n).$$

In particular, we have

$$(a_n, b_n) = (3\iota + \sigma)^{\circ n}(3, 1),$$

where $f^{\circ n}$ denotes the n-fold application of f.

Continuing, because $\iota \circ \sigma = \sigma = \sigma \circ \iota$, we can extend the fact that $(a + b)^p \equiv a^p + b^p \pmod p$ to our context: by the binomial theorem,

$$(3\iota + \sigma)^{\circ p} = \sum_{k=0}^{p} \binom{p}{k} (3\iota)^{\circ k} \sigma^{\circ (p-k)} \equiv (3\iota)^{\circ p} + \sigma^{\circ p} \pmod p.$$

Now, we can compute $(3\iota)^{\circ p} = 3^p \iota \equiv 3\iota \pmod p$, and $\sigma^{\circ 2} = -\iota$ so that $\sigma^{\circ p} = \sigma^{\circ 3} = -\sigma$. Thus,

$$\begin{aligned}(3\iota + \sigma)^{\circ (p+1)} &= (3\iota - \sigma) \circ (3\iota + \sigma) \\ &= (3\iota)^{\circ 2} - \sigma^{\circ 2} \\ &= 10\iota.\end{aligned}$$

Thus, $(a_{p+1}, b_{p+1}) \equiv (10\iota)(3, 1) \equiv (30, 10) \pmod p$, so the answer is $30 \cdot 100 + 10 = \boxed{3010}$.

Solution 2: Fix notation as in the first paragraph of the previous solution. This time around, we compute $(3\iota + \sigma)^{\circ 32}$ by force, using repeated squarings. We have the following computations; all equivalences are $\pmod{31}$.

- $(3\iota + \sigma)^{\circ 2} = (3\iota + \sigma) \circ (3\iota + \sigma) = 9\iota + 6\sigma + \sigma^{\circ 2} = 8\iota + 6\sigma$.
- $(3\iota + \sigma)^{\circ 4} = (8\iota + 6\sigma)^{\circ 2} = 64\iota + 96\sigma + 36\sigma^{\circ 2} \equiv 2\iota + 3\sigma - 5\iota \equiv -3\iota + 3\sigma$.
- $(3\iota + \sigma)^{\circ 8} \equiv (-3\iota + 3\sigma)^{\circ 2} \equiv 9\iota - 18\sigma + 9\sigma^{\circ 2} \equiv -18\sigma \equiv 13\sigma$.
- $(3\iota + \sigma)^{\circ 16} \equiv (13\sigma)^{\circ 2} \equiv 169\sigma^{\circ 2} \equiv 14\sigma^{\circ 2} = -14\iota$.
- $(3\iota + \sigma)^{\circ 32} \equiv (-14\iota)^{\circ 2} \equiv 196\iota \equiv 10\iota$.

Thus, as before we have $(a_{32}, b_{32}) \equiv (30, 10) \pmod{31}$, so the answer is $30 \cdot 100 + 10 = \boxed{3010}$.

9. Lysithea and Felix each have a take-out box, and they want to select among 42 different types of sweets to put in their boxes. They each select an even number of sweets (possibly 0) to put in their box. In each box, there is at most one sweet of any type, although the boxes may have sweets of the same type in common. The total number of sweets they take out is 42. Let N be the number of ways can they select sweets to take out. Compute the remainder when N is divided by $42^2 - 1$.

Answer: 1355

Solution 1: A nice way to find N, the number of ways, is to use generating functions. For generality, let $n = 21$. The number of ways to put each even number of sweets in a box can be represented by the generating function $\sum_{i=0}^{n} \binom{2n}{2i} x^{2i}$, where the coefficient of x^a is the number of ways to put a sweets in the box. Using the roots of unity filter, we can rewrite this as:

$$\frac{1}{2}\left(\sum_{i=0}^{2n}\binom{2n}{i}x^i + \sum_{i=0}^{2n}\binom{2n}{i}(-x)^i\right) = \frac{(1+x)^{2n} + (1-x)^{2n}}{2}.$$

To add the sweets in the two boxes together, we multiply their generating functions together. That is, the generating function representing the sweets in both boxes is

$$\left(\frac{(1+x)^{2n} + (1-x)^{2n}}{2}\right)^2 = \frac{(1+x)^{4n} + 2\left(1-x^2\right)^{2n} + (1-x)^{4n}}{4}.$$

The coefficient of x^{2n} in this generating function is the number of ways in which the total number of sweets is $2n$. Using the binomial expansions, this coefficient is

$$\frac{\binom{4n}{2n} + 2(-1)^n\binom{2n}{n} + \binom{4n}{2n}}{4} = \frac{\binom{4n}{2n} + (-1)^n\binom{2n}{n}}{2}.$$

Plugging in $n = 21$ gives $N = \frac{1}{2}\left(\binom{84}{42} - \binom{42}{21}\right)$.

To find the remainder when N is divided by $42^2 - 1 = 41 \cdot 43$, we note that 41 and 43 are primes, so by the Chinese Remainder Theorem we can compute the residues modulo 41 and 43 separately and combine them at the end. For convenience, we will calculate the residues of $2N = \binom{84}{42} - \binom{42}{21}$.

- Modulo 41, we observe that $\binom{42}{21}$ is a multiple of 41. By Wilson's Theorem, $40! \equiv -1 \pmod{41}$, so

$$\begin{aligned}\binom{84}{42} &\equiv \frac{84!}{42! \cdot 42!} \\ &\equiv \frac{84 \cdot 83 \cdot 82 \cdot 40! \cdot 41 \cdot 40!}{(42 \cdot 41 \cdot 40!)^2} \\ &\equiv \frac{2 \cdot 1 \cdot 2 \cdot (-1) \cdot (-1)}{(1 \cdot (-1))^2} \\ &\equiv 4 \pmod{41}.\end{aligned}$$

  Therefore $2N \equiv 4 \pmod{41}$.

- Modulo 43, we observe that $\binom{84}{42}$ is a multiple of 43. By Wilson's Theorem, $42! \equiv -1 \pmod{43}$. Furthermore, $21! \equiv (-1)^{21} \cdot (42 \cdot 41 \cdots 22) \pmod{43}$, so $(21!)^2 = (-1)^{21} \cdot 42! = 1$

(mod 43). Then

$$\binom{42}{21} \equiv \frac{42!}{21! \cdot 21!} \equiv \frac{-1}{1} \equiv -1 \pmod{43}.$$

Therefore $2N \equiv 1 \pmod{43}$.

Now we combine $2N \equiv 4 \pmod{41}$ and $2N \equiv 1 \pmod{43}$ by the Chinese Remainder Theorem and obtain $2N \equiv 947 \pmod{41 \cdot 43}$. Finally, multiplying both sides by 2^{-1} (namely, we add $41 \cdot 43$ to the right side and divide by 2) gives $N \equiv \boxed{1355} \pmod{41 \cdot 43}$.

Solution 2: It is possible to use a direct counting argument using bijections. Again for generality, let $n = 21$. Number the types of sweets from 1 to $2n$, and let the sets of sweets in the two boxes be A and B. We will consider all combinations of sets A, B for which $|A| + |B| = 2n$. Let us define a function $\phi(A, B)$ as follows: find the sweet of largest value that is present in one box and not the other, and switch the sweet between the boxes. Note that this operation is defined over all sets A, B for which $|A| + |B| = 2n$ and $A \neq B$ (if $A = B$, then there is no sweet that is present in one box and not the other), and that the cardinalities of $|A|$ and $|B|$ change when ϕ is applied. Furthermore, $\phi(\phi(A, B)) = (A, B)$ since we just switch one sweet back and forth, so ϕ is a bijection over the sets in which it is defined, which are all sets A, B for which $|A|+|B| = 2n$ and $A \neq B$.

Note that the number of sets A, B for which $|A| + |B| = 2n$ is $\binom{4n}{2n}$, since both count the number of ways to create a set of $2n$ hats out of $2n$ blue hats and $2n$ red hats. There are $\binom{2n}{n}$ combinations of sets A, B such that $A = B$, so there are $\binom{4n}{2n} - \binom{2n}{n}$ combinations of sets A, B such that $A \neq B$. Among the combinations of sets for which $A \neq B$, since a bijection ϕ exists between combinations for which $|A|$ and $|B|$ are even and those for which $|A|$ and $|B|$ are odd, the number of combinations of sets for which $|A|$ and $|B|$ are even is $\frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}\right)$. So the general formula for the number of ways to select the sweets to take out is $\frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}\right)$ if n is odd, and $\frac{1}{2}\left(\binom{4n}{2n} - \binom{2n}{n}\right) + \binom{2n}{n} = \frac{1}{2}\left(\binom{4n}{2n} + \binom{2n}{n}\right)$ if n is even. Plugging in $n = 21$ gives $N = \frac{1}{2}\left(\binom{84}{42} - \binom{42}{21}\right)$, and the answer extraction proceeds in the same way.

10. Compute the number of integer ordered pairs (a, b) such that $10!$ is a multiple of $a^2 + b^2$.

Answer: 648

Solution: The main idea here is to work in the complex numbers. Indeed, define $\mathbb{Z}[i] = \{a+bi : a, b \in \mathbb{Z}\}$, so we see that $a^2 + b^2 \mid 10!$ if and only if $\alpha = a + bi$ has

$$\alpha\overline{\alpha} \mid 10!,$$

where $\overline{\alpha}$ denotes the complex conjugate. This has the advantage of being a multiplicative divisible condition instead of an additive one, so we will spend the rest of the problem counting the number of possible $\alpha \in \mathbb{Z}[i]$ such that $\alpha\overline{\alpha} \mid 10!$.

We see that we now want to factor $10!$ in $\mathbb{Z}[i]$. To start, we factor $10!$ in $\mathbb{Z}$ by using de Polignac's formula, writing

$$10! = 2^{5+2+1} \cdot 3^{3+1} \cdot 5^2 \cdot 7 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

We can factor some of these further in $\mathbb{Z}[i]$: namely, $2 = (1+i)(1-i) = -i(1+i)^2$ and $5 = (2+i)(2-i)$. This gives the factorization

$$10! = (1+i)^{16} \cdot 3^4 \cdot (2+i)^2(2-i)^2 \cdot 7.$$

It is known but can be checked by hand that all factors above are actually primes in $\mathbb{Z}[i]$, so we have found our prime factorization of 10! in $\mathbb{Z}[i]$.

Now, to finish, we need to divide our given factorization between α and $\overline{\alpha}$ to give $\alpha\overline{\alpha} \mid 10!$. We count the number of possible prime factorizations of $\alpha \mid 10!$.

- For each power of $(1+i)$ dividing α, there must exist a power of $\overline{1+i} = 1-i = -i(1+i)$ dividing $\overline{\alpha}$. In other words, the number of $(1+i)$s dividing α will be equal to the number of $(1+i)$s dividing $\overline{\alpha}$.

  So because there are only 16 powers to go between α and $\overline{\alpha}$, we see α can only be divisible by up to $(1+i)^8$, which totals to 9 options.
- Again, for each power of 3 dividing α, there must exist a power of $\overline{3} = 3$ dividing $\overline{\alpha}$. But again, with only four powers of 3 to go between α and $\overline{\alpha}$, we have that the largest power of 3 available to α is 3^2, which totals to 3 options.
- If a $2 \pm i$ divides α, then we need a $\overline{2 \pm i} = 2 \mp i$ to divide $\overline{\alpha}$, and vice versa. We can more or less count by force, then, the number of ways to distribute the $(2+i)$s and the $(2-i)$s: if α receives $(2+i)^a(2-i)^b$, then $\overline{\alpha}$ will receive $(2+i)^b(2-i)^a$, so we are requiring $a+b \leq 2$ because we only have two powers to go around. In other words, we are looking for triplets of nonnegative integers $(a, b, 2-a-b)$. Since $a+b+(2-a-b) = 2$, a sticks and stones argument tells us that there are $\binom{4}{2} = 6$ such triples.
- If 7 divides α, then $\overline{7} = 7$ divides $\overline{\alpha}$, but 7^2 does not divide 10!. Thus, α cannot have any 7s.

In total, there are $9 \cdot 3 \cdot 6 = 162$ total prime factorizations of α. However, this is not the number of possible elements α because we can add a power of i to the prime factorizations to get different elements. There are four such available powers of i, so there are $4 \cdot 162 = \boxed{648}$ total possible elements α.