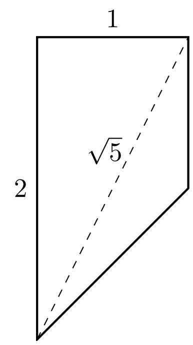
1. To fold a paper airplane, Austin starts with a square paper $FOLD$ with side length 2. First, he folds corners L and D to the square's center. Then, he folds corner F to corner O. What is the longest distance between two corners of the *resulting* figure?

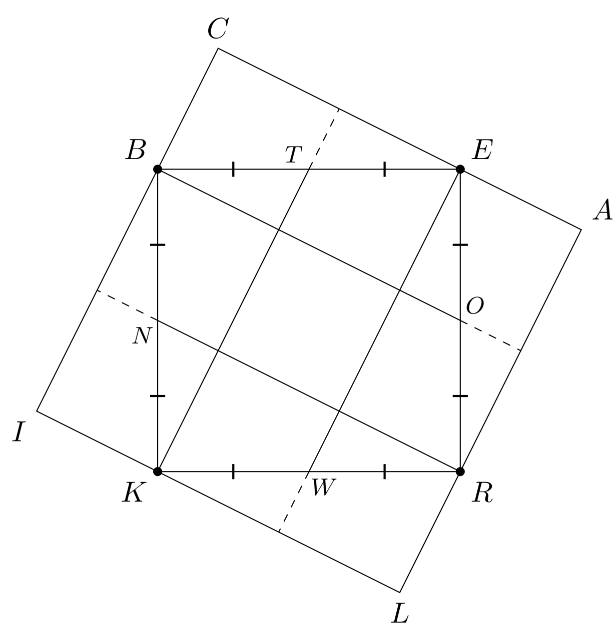
   Answer: $\sqrt{5}$

   Solution:



   After folding, the resulting figure will look like the diagram above. The two corners which are the furthest from each other are the top-right and bottom-left corners. To find this length, we use the Pythagorean theorem: $\sqrt{1^2 + 2^2} = \boxed{\sqrt{5}}$.

2. Sohom constructs a square $BERK$ of side length 10. Darlnim adds points T, O, W, and N, which are the midpoints of $\overline{BE}$, $\overline{ER}$, $\overline{RK}$, and $\overline{KB}$, respectively. Lastly, Sylvia constructs square $CALI$ whose edges contain the vertices of $BERK$, such that $\overline{CA}$ is parallel to $\overline{BO}$. Compute the area of $CALI$.



   Answer: 180

   Solution: Note that

$$[CALI] = [BERK] + [\triangle EAR] + [\triangle RLK] + [\triangle KIB] + [\triangle BCE] = [BERK] + 4[\triangle EAR].$$

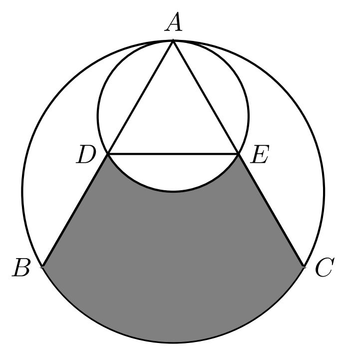
We know $[BERK] = 10^2 = 100$, so it remains to calculate $[EAR]$. Let $\overline{EW}$ and $\overline{RN}$ intersect at X. Since $\angle EAR = 90^\circ$ and $\overline{EA} \parallel \overline{XR}$ and $\overline{EX} \parallel \overline{AR}$, we can conclude that $EXRA$ is a rectangle. Thus, $[\triangle EAR] = [\triangle EXR]$.

Now, note that since $\angle EXR = \angle ERW = 90^\circ$, we have that $\triangle EXR \sim \triangle ERW$. Hence

$$\frac{XR}{ER} = \frac{RW}{EW} \implies \frac{XR}{10} = \frac{5}{\sqrt{5^2 + 10^2}},$$

from which we get $XR = 2\sqrt{5}$. Hence $[\triangle EXR] = \frac{1}{2} \cdot XR \cdot (2 \cdot XR) = 20$, so the answer is $100 + 4(20) = \boxed{180}$.

3. Let equilateral triangle $\triangle ABC$ be inscribed in a circle ω_1 with radius 4. Consider another circle ω_2 with radius 2 internally tangent to ω_1 at A. Let ω_2 intersect sides $\overline{AB}$ and $\overline{AC}$ at D and E, respectively, as shown in the diagram. Compute the area of the shaded region.



Answer: $\mathbf{6\sqrt{3} + 4\pi}$

Solution: Let O_1 and O_2 be the centers of the circles ω_1 and ω_2, respectively. If the foot of the altitude from A to $\overline{DE}$ is F, then we have that $\triangle ADF$ and $\triangle DO_2F$ are both 30-60-90 triangles. Hence $AF = DF\sqrt{3}$ and $O_2F = DF/\sqrt{3}$, so since $AO_2 = 2$ we can compute $DF = \sqrt{3}$, so $AD = AE = 2\sqrt{3}$.

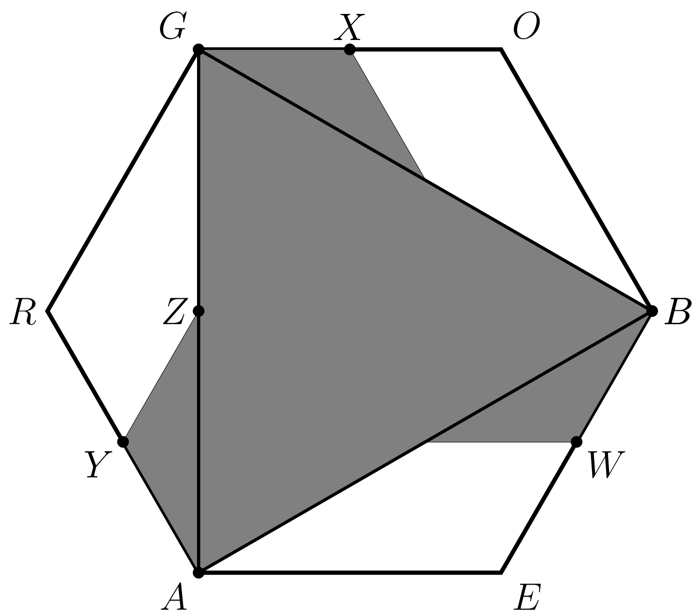
Now, let R_1 be the region bounded by $\overline{AB}$, $\widehat{BC}$, $\overline{CA}$, and let R_2 be the region bounded by $\overline{AD}$, $\widehat{DE}$, and $\overline{EA}$. Since $\triangle ADE \sim \triangle ABC$, the region R_1 is equivalent to the region R_2 dilated from A with ratio $AO_1/AO_2 = 2$. Hence the ratio of their areas is $2^2 = 4$. So, the desired answer is $[R_1] - [R_2] = 3[R_2]$.

Note that $[R_2] = [\triangle ADO_2] + [\triangle AEO_2] + \frac{1}{3}[\omega_2]$, since $\angle DO_2E = 120^\circ$. This can be computed using the lengths from before as $\sqrt{3} + \sqrt{3} + \frac{4\pi}{3}$, so the answer is $3\left(2\sqrt{3} + \frac{4\pi}{3}\right) = \boxed{6\sqrt{3} + 4\pi}$.

4. On regular hexagon $GOBEAR$ with side length 2, bears are initially placed at G, B, A, forming an equilateral triangle. At time $t = 0$, all of them move clockwise along the sides of the hexagon at the same pace, stopping once they have each traveled 1 unit. What is the total area swept out by the triangle formed by the three bears during their journey?

Answer: $\mathbf{\frac{15\sqrt{3}}{4}}$

Solution:



Let X, W, and Y be the midpoints of $\overline{GO}$, $\overline{BE}$, and $\overline{AR}$, respectively. The shaded area in the diagram represents the total area that is swept out as triangle $\triangle GBA$ rotates to triangle $\triangle XWY$. To calculate this area, we will instead subtract the three unshaded regions from the area of the entire hexagon. It suffices to just calculate one, as the other two are equivalent.

Let Z be the intersection of $\overline{GA}$ and $\overline{XY}$. As the two bears move from G to X and from A to Y, they form a line that continuously rotates and shifts from line $\overline{GA}$ to $\overline{XY}$. Thus, the region that is not swept out by this side is quadrilateral $GRYZ$. Note that since X, Y are the midpoints of $\overline{GO}, \overline{RA}$, respectively, we have $\overline{XY} \parallel \overline{GR}$. Thus, $AYZ \sim ARG$, with the ratio of similarity being $\frac{1}{2}$ since Y is the midpoint of $\overline{AR}$. Hence $[\triangle AYZ] = \frac{1}{4}[\triangle ARG]$. Finally, note that we can calculate $AZ = \sqrt{3}$ and $RZ = 1$ from the 30-60-90 triangle $\triangle ARZ$, so $[\triangle ARG] = AZ \cdot RZ = \sqrt{3}$. Thus

$$[GRYZ] = [\triangle ARG] - [\triangle AYZ] = \frac{3}{4}[\triangle ARG] = \frac{3\sqrt{3}}{4}.$$

The area of the entire hexagon can be calculated as $\frac{3(2)^2\sqrt{3}}{2} = 6\sqrt{3}$, so the total area swept out by the three bears is $6\sqrt{3} - 3\left(\frac{3\sqrt{3}}{4}\right) = \boxed{\frac{15\sqrt{3}}{4}}$.

5. Steve has a tricycle which has a front wheel with a radius of 30 cm and back wheels with radii of 10 cm and 9 cm. The axle passing through the centers of the back wheels has a length of 40 cm and is perpendicular to both planes containing the wheels. Since the tricycle is tilted, it goes in a circle as Steve pedals. Steve rides the tricycle until it reaches its original position, so that all of the wheels do not slip or leave the ground. The tires trace out concentric circles on the ground, and the radius of the circle the front wheel traces is the average of the radii of the other two traced circles. Compute the total number of degrees the front wheel rotates. (Express your answer in simplest radical form.)

Answer: $114\sqrt{1601}$

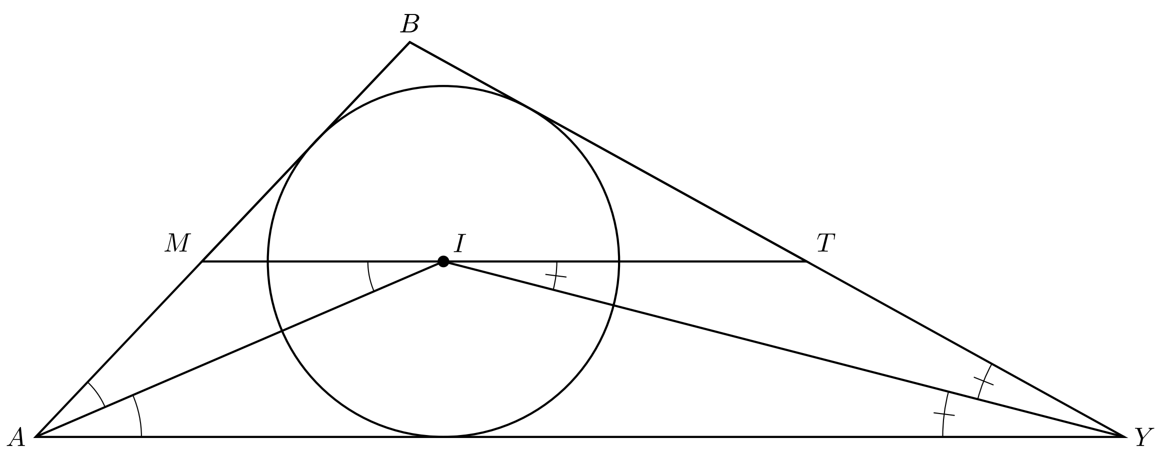
Solution: Note that each of the back wheels is the base of a cone, of which the vertex is the center of the circles traced out by the wheels. For each of the cones, its lateral height is the radius of the circle traced out by the wheel which is the cone's base.

To find the lateral heights, we use similar triangles and the Pythagorean theorem: the frustum whose bases are the two back wheels has a lateral height which is $\frac{1}{10}$ that of the cone formed by the larger back wheel, and we can evaluate that lateral height as $\sqrt{(10-9)^2+40^2} = \sqrt{1601}$, so the lateral height of the cone is $10\sqrt{1601}$. So the radii of the two traced circles from the back wheels is $10\sqrt{1601}$ and $9\sqrt{1601}$, and so the radius of the path the front wheel traces is $\frac{19\sqrt{1601}}{2}$. Since the front wheel has radius 30, the number of rotations it makes is $\frac{19\sqrt{1601}}{60}$, so the number of degrees it rotates is $\frac{19\sqrt{1601}}{60} \cdot 360 = \boxed{114\sqrt{1601}}$.

6. Triangle $\triangle BMT$ has $BM = 4$, $BT = 6$, and $MT = 8$. Point A lies on line $\overleftrightarrow{BM}$ and point Y lies on line $\overleftrightarrow{BT}$ such that $\overline{AY}$ is parallel to $\overline{MT}$ and the center of the circle inscribed in triangle $\triangle BAY$ lies on $\overline{MT}$. Compute AY.

Answer: $\mathbf{\frac{72}{5}}$ **or 14.4**

Solution:



Let I be the incenter of $\triangle BAY$. Since $\overline{AI}$ bisects $\angle YAM$ and $\overline{MI} \parallel \overline{AY}$, we have

$$\angle BAI = \angle YAI = \angle AIM,$$

so $AM = MI$. Likewise, $YT = TI$. By the Angle Bisector Theorem, $MI/IT = MB/BT = 2/3$, so $MI = 16/5$ and $IT = 24/5$. Since $\triangle BMT \sim \triangle BAY$,

$$AY = MT \cdot \frac{BA}{BM} = 8 \cdot \frac{16/5+4}{4} = \boxed{\frac{72}{5}}.$$

7. In triangle $\triangle ABC$ with orthocenter H, the internal angle bisector of $\angle BAC$ intersects $\overline{BC}$ at Y. Given that $AH = 4$, $AY = 6$, and the distance from Y to $\overline{AC}$ is $\sqrt{15}$, compute BC.

Answer: $\mathbf{4\sqrt{35}}$

Solution: Let E denote the foot of the altitude from B to CA. Then

$$AH = \frac{AE}{\sin(\angle AHE)} = \frac{AB\cos(\angle BAC)}{\sin(\angle BCA)} = 2R\cos(\angle BAC),$$

where R denotes the circumradius of ABC and the last equality follows from the extended law of sines. Also, we have that $\sin(\angle BAC/2) = \text{dist}(Y, AC)/AY = \sqrt{15}/6$, so

$$\cos(\angle BAC) = 1 - \left(\frac{\sqrt{15}}{6}\right)^2 = \frac{1}{6}.$$

Since $AH = 4$, we can now compute $R = 12$. So, by the extended law of sines,

$$BC = 2R\sin(\angle BAC) = 24\sqrt{1 - \left(\frac{1}{6}\right)^2} = \boxed{4\sqrt{35}}.$$

8. Anton is playing a game with shapes. He starts with a circle ω_1 of radius 1, and to get a new circle ω_2, he circumscribes a square about ω_1 and then circumscribes circle ω_2 about that square. To get another new circle ω_3, he circumscribes a regular octagon about circle ω_2 and then circumscribes circle ω_3 about that octagon. He continues like this, circumscribing a 2^n-gon about ω_{n-1} and then circumscribing a new circle ω_n about the 2^n-gon. As n increases, the area of ω_n approaches a constant A. Compute A.

Answer: $\boldsymbol{\frac{\pi^3}{4}}$

Solution: Observe that if the radius of ω_{n-1} is r, then after he circumscribes a regular 2^n-gon about the circle and circumscribes a circle about that 2^n-gon, its radius will be $r\sec\left(\frac{\pi}{2^n}\right)$. Hence, the limiting radius of ω_n is the product

$$\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\sec\left(\frac{\pi}{16}\right)\cdots.$$

Consider the partial product

$$\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right).$$

Recalling that $2\sin(x) = \sin(2x)\sec(x)$, we have

$$\begin{aligned}\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right) &= \sin\left(\frac{\pi}{2}\right)\sec\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)\\ &= 2\sin\left(\frac{\pi}{4}\right)\sec\left(\frac{\pi}{8}\right)\cdots\sec\left(\frac{\pi}{2^n}\right)\\ &= \cdots\\ &= 2^{n-1}\sin\left(\frac{\pi}{2^n}\right).\end{aligned}$$

Now observe that $2^{n+1}\sin\left(\frac{\pi}{2^n}\right)$ is the perimeter of a 2^n-gon inscribed in a unit circle, which approaches 2π as n approaches ∞. Hence, the partial product, and hence the radius of ω_n, approaches $\frac{\pi}{2}$ as n approaches ∞. The area of ω_n then approaches $\pi(\pi/2)^2 = \boxed{\frac{\pi^3}{4}}$.

9. Seven spheres are situated in space such that no three centers are collinear, no four centers are coplanar, and every pair of spheres intersect each other at more than one point. For every pair of spheres, the plane on which the intersection of the two spheres lies in is drawn. What is the least possible number of sets of four planes that intersect in at least one point?

Answer: 735

Solution: First, let's see what happens with just three spheres; let their centers be A, B, C and their radii a, b, c. The intersection of two spheres is a circle. If we take a cross-section of the three spheres with the plane ω passing through the three centers, then the three planes in the cross-section are just the radical axes of the three circles of the pairwise intersections of the three spheres. If we let P be the intersection of the radical axes, we have $AP^2 - a^2 = BP^2 - b^2 = CP^2 - c^2$. Then if we take any point Q on the line perpendicular to ω at point P, by the Pythagorean theorem we have $AP^2 + PQ^2 - a^2 = BP^2 + PQ^2 - b^2 = CP^2 + PQ^2 - c^2$, or $AQ^2 - a^2 = BQ^2 - b^2 = CQ^2 - c^2$, so this point also lies on all three planes. Since each intersection of two of the three planes is already a line, we deduce that the intersection of all three planes, which is the "radical axis" of the spheres, is a line.

We can go further: if we add a fourth sphere D with radius d, any set of three of the four spheres defines a "radical axis". If we take two different sets of three spheres, they have two spheres in common, so the two "radical axes" defined by the two sets of spheres must lie on a common plane. Since they cannot be parallel, they must intersect at a point R, where it must be the case that $AR^2 - a^2 = BR^2 - b^2 = CR^2 - c^2 = DR^2 - d^2$. We can call this the "radical center" of the four spheres, and all six planes on which the pairwise intersections of the four spheres lie all pass through this point.

So, the sets of four planes which must all intersect at a single point must include those which have three planes coming from the pairwise intersections of three spheres, and those which have all four planes coming from the pairwise intersections of the same four spheres. Since no three centers are collinear and no four centers are coplanar, every plane drawn will intersect a "radical axis" of three spheres. It can then be shown that there exists a configuration of spheres such that these are the only such sets of four planes.

The number of ways to choose three spheres, and thus three planes which intersect at a line, is $\binom{7}{3} = 35$, and the number of ways to choose a different plane to intersect this line is $\binom{7}{2} - 3 = 18$, for a total of $35 \cdot 18 = 630$ ways for this case. In the case that the four planes do not have a triple whose intersection is a line, we choose four spheres, and then pick four out of the six planes coming fron the pairwise intersections where no three of them come from the same pairwise intersection of three spheres. The number of ways to choose four spheres is $\binom{7}{4} = 35$, and the number of ways to choose four of the six planes with the restriction is $\binom{6}{4} - 4 \cdot 3 = 3$, for a total of $35 \cdot 3 = 105$ ways for this case. So, the least possible number of sets of four planes with a common point is $630 + 105 = \boxed{735}$.

10. In triangle $\triangle ABC$, E and F are the feet of the altitudes from B to $\overline{AC}$ and C to $\overline{AB}$, respectively. Line $\overleftrightarrow{BC}$ and the line through A tangent to the circumcircle of ABC intersect at X. Let Y be the intersection of line $\overleftrightarrow{EF}$ and the line through A parallel to $\overline{BC}$. If $XB = 4$, $BC = 8$, and $EF = 4\sqrt{3}$, compute XY.

Answer: $\mathbf{2\sqrt{21}}$

Solution: We claim that $\overleftrightarrow{XY}$ is the radical axis of point-circle A and circle $(BFEC)$. To prove this, first let H be the orthocenter of ABC; then, we have that $AFHE$ and $BFEC$ are cyclic quadrilaterals, since $\angle AFH = \angle AEH = 90^\circ$ and $\angle BFC = \angle BEC = 90^\circ$.

Now, note that $\angle YAC = \angle ACB = \angle AFE$ (where the last equality can be seen from $BFEC$ being cyclic), so $\overline{YA}$ is tangent to $(AFHE)$. Hence by Power of a Point on this circle, $YA^2 = YE \cdot YF$. But note that YA^2 is the power of Y with respect to the point-circle A and $YE \cdot YF$ is the power of Y with respect to $(BFEC)$, so Y lies on the radical axis of (A) and $(BFEC)$.

Similarly, since $\overline{XA}$ is tangent to (ABC), by Power of a Point of X with respect to (ABC), we have that $XA^2 = XB \cdot XC$. But XA^2 is the power of X with respect to the point-circle A and $XB \cdot XC$ is the power of X with respect to $(BFEC)$, so X lies on the radical axis of (A) and $(BFEC)$. Hence we have proven the claim.

Now, note that the center of (A) is simply A itself, and since $\angle BFC = 90°$, $\overline{BC}$ is a diameter of $(BFEC)$, so that the center of this circle is the midpoint of $\overline{BC}$. Let this point be M. Then, we have that $\overline{XY} \perp \overline{AM}$ since the radical axis of two circles is perpendicular to the line through their centers.

Define $T = XY \cap AM$. Since $\angle ATX = 90°$, by the Pythagorean Theorem we have

$$AX^2 - TX^2 = AT^2 = AY^2 - TY^2.$$

Also, since $\overline{AY} \parallel \overline{CX}$, we have $\triangle ATY \sim \triangle MTX$, so $\frac{TX}{TY} = \frac{MX}{AY}$. Now, by Power of a Point, we compute $AX = \sqrt{XB \cdot XC} = 4\sqrt{3}$. Also, note that

$$\angle AFY = \angle ACX, \qquad \angle FAE = \angle CAB, \qquad \angle EAY = \angle BAX.$$

Hence we have that $AFEY \sim ACBX$; in particular, $\frac{AY}{AX} = \frac{EF}{BC} = \frac{\sqrt{3}}{2}$. Hence $AY = 4\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6$. Finally, note that $MX = XB + BC/2 = 8$. Now, plugging these values into the equations $AX^2 - TX^2 = AY^2 - TY^2$ and $\frac{TX}{TY} = \frac{MX}{AY}$ gives

$$\left.\begin{array}{c} 48 - TX^2 = 36 - TY^2 \\ \dfrac{TX}{TY} = \dfrac{4}{3} \end{array}\right\} \implies TX = \frac{8\sqrt{21}}{7}, TY = \frac{6\sqrt{21}}{7}.$$

Hence $XY = TX + TY = \boxed{2\sqrt{21}}$.