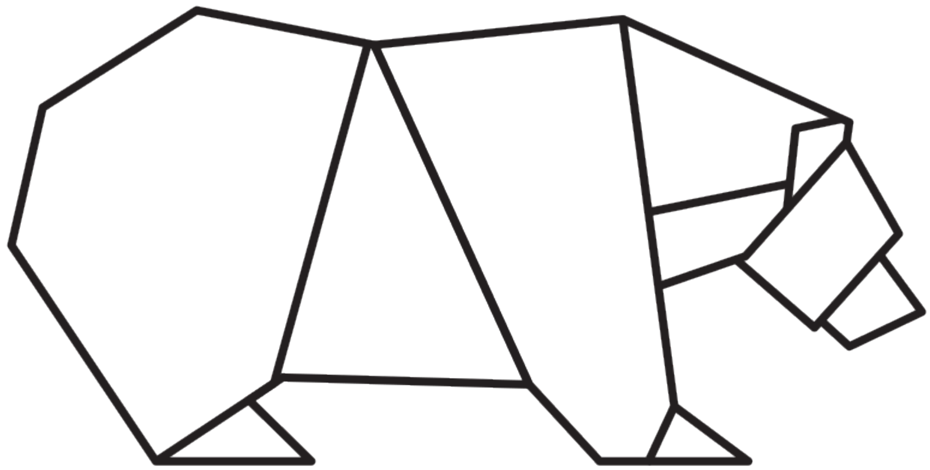
Berkeley Math Tournament 2024

Algebra Test



November 2, 2024

Time limit: 60 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators. Protractors, rulers, and compasses are permitted.

Answer format overview:

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2}+\frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π, e, $\sin(10^\circ)$, and $\ln 2$ in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

DO NOT DISCUSS OR DISTRIBUTE ONLINE UNTIL DECEMBER 8, 2024

1. At a certain point in time, Nikhil had 3 more apples than Brian. Theo then gave Nikhil 9 apples and took away 3 apples from Brian. Now, Nikhil has twice as many apples as Brian. Compute the number of apples that Nikhil and Brian now have in total.

2. Find the real number x satisfying
$$\frac{x^2-20}{x^2+20x+4}=\frac{x^2-24}{x^2+24x+4}=\frac{1}{2}.$$

3. Suppose $a_1, a_2, \ldots$ is an arithmetic sequence, and suppose $g_1, g_2, \ldots$ is a geometric sequence with common ratio 2. Suppose $a_1+g_1=1$ and $a_2+g_2=1$. If $a_{24}=g_7$, find a_{2024}.

4. For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n and let $\lceil n \rceil$ be the smallest integer greater than or equal to n. For example, $\lfloor 2.5 \rfloor = 2$ and $\lfloor 2 \rfloor = 2$, while $\lceil 2.5 \rceil = 3$ and $\lceil 2 \rceil = 2$. Find the greatest integer x such that $\left\lfloor \frac{x}{20}+20 \right\rfloor = \left\lceil \frac{x}{24}+24 \right\rceil$.

5. An ordered pair (a, b) of real numbers is Z-*nice* if x^3+ax+b has 3 distinct roots p, q, r such that $|p-2024|=|q-2024|=|r-2024|=Z$. Find the greatest possible real value of Z such that there is exactly one Z-nice ordered pair.

6. There exist nonzero real numbers B, M, and T that satisfy the equations:
$$2B+M+T-2B^2-2BM-2MT-2BT=0,$$
$$B+2M+T-3M^2-3BM-3MT-3BT=0,$$
$$B+M+2T-4T^2-4BM-4MT-4BT=0.$$
Compute $2B+3M+4T$.

7. Compute the number of positive integer triples (B, M, T) satisfying $B, M, T < 24$ and
$$BM+MT+BT=(B+M+T)\sqrt[3]{BMT}.$$

8. Let α be a positive real number. Over all choices of positive real numbers w, x, y, z satisfying
$$wx+yz=\alpha,$$
$$wy+xz=\alpha,$$
$$wz+xy=\alpha,$$
the minimum value of $w+2x+3y+4z$ is $\frac{\alpha}{2}$. Compute α.

9. Define two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, such that $a_0=b_0=\sqrt{3}$ and for $n \geq 0$:
$$a_{n+1}=a_n+\sqrt{1+a_n^2}$$
$$b_{n+1}=\frac{b_n}{1+\sqrt{1+b_n^2}}.$$
Find the smallest real number M such that $M > |a_i b_i - a_j b_j|$ for any integers $i, j > 0$.

10. Let $\omega = e^{\frac{2\pi i}{5}}$, where $i=\sqrt{-1}$. If
$$S=\prod_{0\leq a,b,c<5}\left(\omega^a-\sec\left(\frac{2\pi(b-a)}{5}\right)\cdot\omega^b+\omega^c\right),$$
compute the remainder when S is divided by 101.